New techniques in calculation of sutured instanton Floer homology

by Heegaard diagrams, Euler characteristics, and Dehn surgery formulae

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This dissertation is submitted for the degree of
Doctor of Philosophy

Churchill College

April 2022

Declaration

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Abstract

Kronheimer-Mrowka conjectured that sutured instanton Floer homology $SHI(M,\gamma)$ has the same dimension as the sutured Floer homology $SFH(M,\gamma)$ constructed by Juhász for any balanced sutured manifold (M,γ). Motivated by their conjecture, we introduce new techniques for calculations of sutured instanton Floer homology, some of which are inspired by analogous results in Heegaard Floer theory.

The first technique is based on Heegaard diagrams of balanced sutured manifolds, from which we obtain an upper bound on the dimension of SHI. For any rationally null-homologous knot K in a closed 3-manifold Y, we prove the dimension of the instanton knot homology $KHI(Y,K)$ is greater than or equal to the dimension of the framed instanton homology $I^{\sharp}(Y)$. We also use this technique to compute the instanton knot homology of $(1,1)$-knots that are also L-space knots. In particular, we calculate the homologies for all torus knots in S^3.

The second technique is based on the identification of Euler characteristics of SFH and SHI, from which we obtain a lower bound on the dimension of SHI. We construct a decomposition of SHI analogous to the spin^c structure decomposition of SFH, and prove that the enhanced Euler characteristic defined by this decomposition equals to the Euler characteristic of SFH. We introduce a family of $(1,1)$-knots called **constrained knots** and show that the upper bound from the first technique coincides with the lower bound from the second technique.

The third technique relates $KHI(S^3,K)$ to $I^{\sharp}(S^3_n(K))$ by a large surgery formula, where $S^3_n(K)$ is obtained from a knot $K\subset S^3$ by n-Dehn surgery. As an application, we show that $S^3_r(K)$ admits an irreducible SU(2) representation for a dense set of slopes r unless K is a prime knot and the coefficients of the Alexander polynomial $\Delta_K(t)$ lie in $\{-1,0,1\}$. In particular, any hyperbolic alternating knot satisfies this property.

Acknowledgements

I would like to thank my supervisor Jacob Rasmussen for introducing me many interesting topics in this dissertation, including the relation between various sutured Floer homologies and the construction of constrained knots. I'm deeply grateful of his patient guidance and helpful advice during my Ph.D life. I would also like to thank my collaborators and friends Zhenkun Li, John A. Baldwin, and Steven Sivek for sharing their insightful thoughts and surprising ideas during my research on sutured instanton homology and related topics. I hope our fruitful collaboration will continue in the future. I am also grateful to the extremely patient examiners Ailsa Keating and Steven Sivek who check this dissertation.

I would like to thank Yi Liu for inviting me to BICMR, Peking University and having many inspiring conversations with me. I would also like to thank Nathan M. Dunfield, Peter Kronheimer, Jianfeng Lin, Ciprian Manolescu, Tomasz Mrowka, Clifford Taubes, Donghao Wang, Yi Xie, Ian Zemke, and Boyu Zhang for expanding my horizons on gauge theory and low-dimensional topology.

I would like to thank my parents and relatives for their support and constant encouragement. I would also like to thank my best friends Chestnut, Kunkun, Dunjieshe, Sirius, Marmot, Fzzzhang, Shengge, Xuange for their company and enlightenment. I am also grateful to Wenfeng Chen, Yujia Lu, Shengyu Zou, Ruide Fu, Longke Tang, Muge Chen, Zhengyang Cai, Zhaowei Tao, Qiuhao Wang, Liqiang Huang, Jiejin Deng, Chunlei Liu, Minghao Miao, Liuyun Yang for impressive conversations. I enjoy the time in kalakala when I was writing the first draft of this dissertation.

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Chapter 1

Introduction

This dissertation studies gauge theoretical invariants for 3-manifolds (possibly with extra data) with a topological and algebraic approach. The main objects are balanced sutured manifolds defined as follows.

Definition 1.0.1 ([Juh06, Definition 2.2])**.** A **balanced sutured manifold** (M,γ) consists of a compact oriented 3-manifold M with non-empty boundary together with a closed 1-submanifold γ on ∂M called the **suture**. Let $A(\gamma) = [-1,1]\times\gamma$ be an annular neighborhood of $\gamma \subset \partial M$ and let $R(\gamma) = \partial M\backslash\text{int}(A(\gamma))$. There are required to satisfy the following properties.

(1) Neither M nor $R(\gamma)$ has a closed component.

(2) If $\partial A(\gamma) = -\partial R(\gamma)$ is oriented in the same way as γ, then we require that this orientation of $\partial R(\gamma)$ induces the orientation on $R(\gamma)$, which is called the **canonical orientation**.

(3) Let $R_+(\gamma)$ be the part of $R(\gamma)$ for which the canonical orientation coincides with the induced orientation on ∂M from M, and let $R_-(\gamma) = R(\gamma)\backslash R_+(\gamma)$. We require that the Euler characteristics of $R_\pm(\gamma)$ are equal, *i.e.*, $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$. If γ is clear in the context, we simply write $R_\pm = R_\pm(\gamma)$, respectively.

Example 1.0.2. Suppose Y is a closed 3-manifold and $K \subset Y$ is a knot. Let $Y(1)$ be obtained from Y by removing a 3-ball and let δ be a simple closed curve on $\partial Y(1)$. Let $Y\backslash K := Y - \text{int}N(K)$ be the knot complement ($\partial Y\backslash K \cong T^2$) and let $\gamma_K \subset \partial Y\backslash K$ consist of two meridians of K with opposite orientations. Then both $(Y(1),\delta)$ and $(Y\backslash K,\gamma_K)$ are balanced sutured manifolds.

Sutured manifolds were first introduced by Gabai [Gab83, Gab87a, Gab87b], for which the suture is more flexible. For a balanced sutured manifold, there is no torus suture and the surfaces $R_\pm$ have the same genus. For such family of sutured manifolds, Juhász [Juh06] constructed invariants called **sutured (Heegaard) Floer homology**, which is denoted by $SFH(M,\gamma)$. Originally it was a $\mathbb{Z}$-module. For simplicity, we also consider it as a vector space over $\mathbb{F}_2$ in this dissertation, where $\mathbb{F}_2$ is the field with two elements. The construction is based on Heegaard Floer theory, which was started by Ozsváth-Szabó [OS04d]. For closed manifolds and knots, sutured Floer homology recovers hat versions of invariants in Heegaard Floer theory. For the balanced sutured manifolds $(Y(1),\delta)$ and $(Y\backslash K,\gamma_K)$ in Example 1.0.2, there are canonical isomorphisms

$$SFH(Y(1),\delta)\cong\widehat{HF}(Y) \text{ and } SFH(Y(K),\gamma)\cong\widehat{HFK}(Y,K), \tag{1.0.1}$$

where $\widehat{HF}$ is Heegaard Floer homology (*c.f.* Ozsváth-Szabó [OS04d]) and $\widehat{HFK}$ is knot Floer homology (*c.f.* Ozsváth-Szabó [OS04b] and Rasmussen [Ras03]).

Later, Kronheimer-Mrowka made analogous constructions for balanced sutured manifolds in monopole theory and instanton theory [KM10b], which are called **sutured monopole Floer homology** and **sutured instanton Floer homology**. These invariants are denoted by $SHM(M,\gamma)$ and $SHI(M,\gamma)$, which are a $\mathbb{Z}$-module (or a module over a Novikov ring) and a $\mathbb{C}$-vector space, respectively. The idea of the construction is to embed the balanced sutured manifold into a closed 3-manifold Y, and then consider submodules or subspaces of the monopole Floer homology (*c.f.* Kronheimer-Mrowka [KM07]) and the instanton Floer homology (*c.f.* Floer [Flo88, Flo90]) of Y.

Inspired by the isomorphisms in (1.0.2), Kronheimer-Mrowka [KM10b] defined gauge theoretical invariants for closed 3-manifolds and knots as follows.

$$\widetilde{HM}(Y):=SHM(Y(1),\delta) \text{ and } KHM(Y,K):=SHM(Y\backslash K,\gamma_K),$$

$$I^\sharp(Y):=SHI(Y(1),\delta) \text{ and } KHI(Y,K):=SHI(Y\backslash K,\gamma_K).$$

Note that the notations $\widetilde{HM}(Y)$ and $I^\sharp(Y)$ were first used by Bloom [Blo09] and Kronheimer-Mrowka [KM11] for other constructions, which are essentially isomorphic to $SHM(Y(1),\delta)$ and $SHI(Y(1),\delta)$, respectively. We call $I^\sharp(Y)$ the **framed instanton homology** of Y and $KHI(Y,K)$ the **instanton knot homology** of (Y,K).

It is an interesting question to study the relationship among SFH, SHM, and SHI. In this line, Lekili [Lek13] and Baldwin-Sivek [BS21c] proved the first two invariants are isomorphic (with the same coefficients) for any balanced sutured manifolds. Their proofs depend on the isomorphism of Heegaard Floer homology and monopole Floer homology for closed 3-manifolds by Kutluhan-Lee-Taubes [KLT20], or Taubes [Tau10] combined with

Colin-Ghiggini-Honda [CGH17]. The relation with SHI is still open. The results in this dissertation are motivated by the following conjecture due to Kronheimer-Mrowka.

Conjecture 1.0.3 ([KM10b])**.** For a balanced sutured manifold (M,γ), we have

$$SHI(M,\gamma) \cong SFH(M,\gamma)\otimes\mathbb{C}.$$

In particular, for a knot K in a closed 3-manifold Y, we have

$$I^{\sharp}(Y) \cong \widehat{HF}(Y)\otimes\mathbb{C} \text{ and } KHI(Y,K) = \widehat{HFK}(Y,K)\otimes\mathbb{C}.$$

Here homologies in Heegaard Floer homology are considered as $\mathbb{Z}$-modules.

In general, (sutured) instanton Floer homology is hard to calculate since the construction involves solutions of PDEs. Some examples were calculated by groups of people [Sca15, SS18, LPCS20, BS21a, ABDS20]. However, if we choose a good Heegaard diagram of a given (sutured or closed) manifold, its Heegaard Floer homology can be easily calculated [SW10, MOT09, OSS15]. Moreover, Lipshitz-Ozsváth-Thurston [LOT18] extended Heegaard Floer theory to bordered 3-manifolds (called bordered Floer homology) and provided an algorithm to calculate the hat version of Heegaard Floer homology. For a 3-manifold with torus boundary, Hanselman-Rasmussen-Watson [HRW17, HRW18] proposed a geometric and graphical way to understand the algebraic structure of the bordered Floer homology.

In this dissertation, we introduce new techniques for calculations of sutured instanton Floer homology, some of which are inspired by analogous results in Heegaard Floer theory. These techniques are based on Heegaard diagrams of (sutured) manifolds, various versions of Euler characteristics of sutured instanton Floer homology, and formulae relating the $KHI(S^3,K)$ and $I^{\sharp}(S^3_n(K))$, where $S^3_n(K)$ is obtained from K by n-Dehn surgery. We introduce the results in the following three sections.

1.1 Calculation by Heegaard diagrams

The first technique to calculate sutured instanton Floer homology is based on Heegaard diagrams, from which we obtain an upper bound on the dimension. The results in this section are based on [LY22].

Theorem 1.1.1. *Suppose Y is a rational homology sphere, and $K\subset Y$ is a knot. Suppose $(\Sigma,\alpha,\beta,z,w)$ is a doubly-pointed Heegaard diagram of (Y,K). Then there is a balanced sutured handlebody (H,γ) constructed from $(\Sigma,\alpha,\beta,z,w)$ (c.f. Subsection 3.2.1), so that the* *following hold*

$$\dim_{\mathbb{C}} I^{\sharp}(-Y) \leq \dim_{\mathbb{C}} KHI(-Y,K) \leq \dim_{\mathbb{C}} SHI(-H,-\gamma).$$

Remark 1.1.2. For most arguments in this dissertation, there are minus signs before the manifold and the suture, which means that we take the reverse orientation. This is because the proofs are based on contact gluing maps for sutured instanton Floer homology (*c.f.* Subsection 2.3.4).

The proof of Theorem 1.1.1 makes use of rationally null-homologous tangles in balanced sutured manifolds. In particular, we proved the following proposition.

Proposition 1.1.3. *Suppose (M,γ) is a balanced sutured manifold and T is a connected vertical tangle in (M,γ) (c.f. Definition 3.1.1). Suppose $M_T = M\backslash N(T)$ and $\gamma_T = \gamma \cup m_T$, where m_T is the meridian of T. If $[T] = 0 \in H_1(M,\partial M;\mathbb{Q})$, then we have*

$$\dim_{\mathbb{C}} SHI(-M,-\gamma) \leq \dim_{\mathbb{C}} SHI(-M_T,-\gamma_T).$$

By Proposition 1.1.3, we also prove a generalization of the first inequality in Theorem 1.1.1, which generalizes the result for null-homologous knots by Wang [Wan20, Proposition 1.18].

Proposition 1.1.4. *Suppose Y is a closed 3-manifold and $K \subset Y$ is a knot such that*

$$[K] = 0 \in H_1(Y;\mathbb{Q}).$$

Then we have

$$\dim_{\mathbb{C}} I^{\sharp}(-Y) \leq \dim_{\mathbb{C}} KHI(-Y,K).$$

In Theorem 1.1.1, we bound the dimensions of $I^{\sharp}(-Y)$ and $KHI(-Y,K)$ by the dimension of sutured instanton Floer homology $SHI(-H,-\gamma)$, which is still difficult to compute in general. However, in the case where H is a handlebody, an upper bound on $\dim_{\mathbb{C}} SHI(H,\gamma)$ can be calculated via bypass exact triangles (for bypass exact triangle, *c.f.* [BS22, Theorem 1.21], and for the algorithm to obtain an upper bound, *c.f.* [GL19, Section 4]). In particular, we apply this theorem to $(1,1)$-knots in lens spaces, whose Heegaard diagrams can be described explicitly (*c.f.* Proposition 3.2.14), and obtain the following theorem.

Theorem 1.1.5. *Suppose Y is a lens space, and $K \subset Y$ is a $(1,1)$-knot. Then we have*

$$\dim_{\mathbb{C}} KHI(Y,K) \leq \dim_{\mathbb{F}_2} \widehat{HFK}(Y,K).$$

Prior to the current paper, there are two main approaches to estimate the dimension of KHI. The first is via the spectral sequence from Khovanov homology to instanton knot homology established by Kronheimer-Mrowka [KM11]. This bound is sharp for all alternating knots and many other knots. However, Khovanov homology is only defined for knots in S^3, so we cannot have any information for knots in other 3-manifolds. The second way is to study a set of explicit generators of the instanton knot homology and its variances for some special families of knots, and the number of generators bounds the dimension of homology. This idea has been exploited by Hedden-Herald-Kirk [HHK14] and Daemi and Scaduto [DS19]. Our Theorem 1.1.1 and Theorem 1.1.5 then offer a totally new way to obtain an upper bound on $\dim_{\mathbb{C}} KHI$, and the following corollary indicates that this bound is sharp for many examples.

Corollary 1.1.6. *Suppose $K \subset S^3$ is a $(1,1)$-knot that is also an L-space knot. Then*

$$\dim_{\mathbb{C}} KHI(S^3,K) = \dim_{\mathbb{F}_2} \widehat{HFK}(S^3,K).$$

Remark 1.1.7. Recall that a closed 3-manifold Y is called a **(Heegaard Floer) L-space** if Y is a rational homology sphere and

$$\dim_{\mathbb{F}_2} \widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|$$

and a knot $K \subset Y$ is called a **(Heegaard Floer) L-space knot** if Y is an L-space and some nontrivial surgery on K also gives an L-space. Note that lens spaces (including S^3) are all L-spaces.

Proof of Corollary 1.1.6. Suppose the Alexander polynomial of K is $\Delta_K(t) = \sum_{i\in\mathbb{Z}} c_i t^i$. From Ozsváth-Szabó [OS05b, Theorem 1.2], we have

$$\dim_{\mathbb{F}_2} \widehat{HFK}(S^3,K) = \sum_{i\in\mathbb{Z}} |c_i|.$$

In instanton theory, the main result of Kronheimer and Mrowka [KM10a], or Lim [Lim10], states that the Euler characteristic of the i-th grading of $KHI(S^3,K)$ equals $\pm c_i$. As a result, we have

$$\dim_{\mathbb{C}} KHI(S^3,K) \geq \sum_{i\in\mathbb{Z}} |c_i|.$$

Hence Theorem 1.1.5 applies and we conclude the desired equality. □

Corollary 1.1.6 would provide many examples whose related spectral sequences from Khovanov homology to instanton knot homology have some nontrivial intermediate pages.

In particular, for torus knots, previously there were only partial computations of KHI from the related spectral sequences (*c.f.* [KM14, LZ20]; see also [HHK14] for another approach to obtaining upper bounds from generators), while Corollary 1.1.6 applies to torus knots directly since torus knots admit lens spaces surgeries (*c.f.* Moser [Mos71]).

Corollary 1.1.8. *For a torus knot $K = T_{(p,q)}$, we write its Alexander polynomial as*

$$\Delta_K(t) = t^{-\frac{(p-1)(q-1)}{2}} \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)} = \sum_{i=-\frac{(p-1)(q-1)}{2}}^{\frac{(p-1)(q-1)}{2}} c_i t^i.$$

Then we have

$$\dim_{\mathbb{C}} KHI(S^3, K, i) = |c_i|,$$

where i denotes the Alexander grading of $KHI(S^3, K)$.

1.2 Calculation by Euler characteristics

The second technique to calculate sutured instanton Floer homology is based on its Euler characteristic, from which we obtain a lower bound on the dimension. Note that there is a relative $\mathbb{Z}_2$-grading on sutured instanton Floer homology so that we can take the Euler characteristic up to sign. The results in this section are based on [LY21b, LY21a, Ye21].

Theorem 1.2.1. *Suppose (M, γ) is a balanced sutured manifold and $H = H_1(M; \mathbb{Z})$. Then there is a (possibly noncanonical) decomposition*

$$SHI(M, \gamma) = \bigoplus_{h \in H} SHI(M, \gamma, h).$$

This decomposition depends on some auxiliary choices. We define the ***enhanced Euler characteristic*** *of SHI by*

$$\chi_{\rm en}(SHI(M,\gamma)) := \sum_{h\in H} \chi(SHI(M,\gamma,h)) \cdot h \in \mathbb{Z}[H]/\pm H.$$

Then we have

$$\chi_{\rm en}(SHI(M,\gamma)) = \chi(SFH(M,\gamma)) \in \mathbb{Z}[H]/\pm H. \tag{1.2.1}$$

The decomposition in Theorem 1.2.1 is motivated by the following spin^c decomposition

$$SFH(M,\gamma) = \bigoplus_{\mathfrak{s}\in {\rm Spin}^c(M,\gamma)} SFH(M,\gamma,\mathfrak{s}).$$

Note that $\mathrm{Spin}^c(M,\gamma)$ is an affine space over $H^2(M,\partial M;\mathbb{Z}) \cong H_1(M;\mathbb{Z}) = H$. Fixing a spin^c structure $\mathfrak{s}_0$, we define

$$\chi(SFH(M,\gamma)) := \sum_{\mathfrak{s}\in \mathrm{Spin}^c(M,\gamma)} \chi(SFH(M,\gamma,\mathfrak{s}))\cdot \mathrm{PD}(\mathfrak{s}-\mathfrak{s}_0) \in \mathbb{Z}[H]/\pm H, \tag{1.2.2}$$

where PD is the Poincaré duality map. Hence, though the decomposition in Theorem 1.2.1 has not been proved to be canonical, we expect it to be well-defined up to a global grading shift of H.

If $H_1(M;\mathbb{Z})$ has no torsion, then Theorem 1.2.1 reduces to the following case, which will be proved first in Chapter 4.

Theorem 1.2.2. *Suppose (M,γ) is a balanced sutured manifold and $S_1,\dots,S_n$ are properly embedded admissible surfaces (c.f. Definition 2.3.19) generating $H_2(M,\partial M)/\mathrm{Tors}$. Then there exist well-defined $\mathbb{Z}^n$-gradings on $SHI(M,\gamma)$ and $SFH(M,\gamma)$ induced by these surfaces. Equivalently, we have*

$$SHI(M,\gamma) = \bigoplus_{(i_1,\dots,i_n)\in\mathbb{Z}^n} SHI(M,\gamma,(S_1,\dots,S_n),(i_1,\dots,i_n))$$

*and an analogous result holds for $SFH(M,\gamma)$. We define the **graded Euler characteristic** by*

$$\chi_{\mathrm{gr}}(SHI(M,\gamma)) := \sum_{(i_1,\dots,i_n)\in\mathbb{Z}^n} \chi(SHI(M,\gamma,(S_1,\dots,S_n),(i_1,\dots,i_n)))\cdot t_1^{i_1}\cdots t_n^{i_n}, \tag{1.2.3}$$

and define $\chi_{\mathrm{gr}}(SFH(M,\gamma))$ similarly. Then we have

$$\chi_{\mathrm{gr}}(SHI(M,\gamma)) \sim \chi_{\mathrm{gr}}(SFH(M,\gamma)),$$

where $\sim$ means two polynomials are equal up to multiplication by $\pm t_1^{j_1}\cdots t_n^{j_n}$ for some $(j_1,\dots,j_n)\in\mathbb{Z}^n$,

Remark 1.2.3. Suppose that $t_1,\dots,t_n$ represent generators of

$$H' = H_1(M;\mathbb{Z})/\mathrm{Tors} \cong H_2(M,\partial M;\mathbb{Z})/\mathrm{Tors}.$$

Then $\sim$ means the equality holds for elements in $\mathbb{Z}[H']/\pm H'$. The graded Euler characteristic $\chi_{\mathrm{gr}}(SFH(M,\gamma))$ is just the image of $\chi(SFH(M,\gamma))$ under the map

$$\mathbb{Z}[H_1(M;\mathbb{Z})] \to \mathbb{Z}[H_1(M;\mathbb{Z})/\mathrm{Tors}].$$

The Euler characteristic $\chi(SFH(M,\gamma))$ was studied by Friedl-Juhász-Rasmussen [FJR09]. Explicitly, we have

$$\chi(SFH(M,\gamma)) = \tau(M,\gamma),$$

where $\tau(M,\gamma)$ is a Turaev-type torsion element that can be calculated by Fox calculus. In particular, if ∂M consists of tori and γ consists of two parallel copies of a curve m_i with opposite orientations on each boundary component, by the proof of [FJR09, Lemma 6.1] and [RR17, Proposition 2.1], we have

$$\tau(M,\gamma) = \tau(M)\cdot\prod_i([m_i]-1),$$

where $\tau(M)$ is the Turaev torsion of M [Tur02]. When M is the complement of a knot K in S^3, then

$$\tau(M) = \frac{\Delta_K(t)}{t-1}.$$

When M is the complement of a link L in S^3 of more than one component, then

$$\tau(M) = \Delta_L(t_1,\ldots,t_n),$$

where the right hand side is the multivariable Alexander polynomial of L. Then we have the following corollaries.

Corollary 1.2.4. *Suppose K is a knot in a closed oriented 3-manifold Y and suppose M is the knot complement. Let $[m]\in H = H_1(M;\mathbb{Z})$ be the homology class of the meridian of K. Then we have*

$$\chi_{\mathrm{en}}(KHI(Y,K)) = \tau(M)\cdot([m]-1)\in\mathbb{Z}[H]/\pm H.$$

Remark 1.2.5. Analogous results of Corollary 1.2.4 in Heegaard Floer theory can be found in [RR17, Proposition 2.1] and [Ras07, Proposition 3.1]. Also, Corollary 1.2.4 is a generalization of work of Lim [Lim10] and Kronheimer-Mrowka [KM10a], in which they proved the same results only for knots inside S^3.

Corollary 1.2.6. *Suppose M is a compact manifold whose boundary consists of tori $T_1,\ldots,T_n$. Suppose*

$$\gamma = \bigcup_{j=1}^{n} m_j\cup(-m_j)$$

consists of two simple closed curves with opposite orientations on each torus. Suppose $H = H_1(M;\mathbb{Z})$ *and* $[m_1],\dots,[m_n]$ *are homology classes. Then we have*

$$\chi_{\mathrm{en}}(SHI(M,\gamma)) = \tau(M)\cdot\prod_{j=1}^{n}([m_j]-1) \in \mathbb{Z}[H]/\pm H. \tag{1.2.4}$$

In particular, suppose $L \subset S^3$ *is an n-component link with* $n \geq 2$*. Let* $(i_1,\dots,i_n)$ *denote the* $\mathbb{Z}^n$*-grading on* $KHI(L)$ *induced by Seifert surfaces of components of L. Then we have*

$$\chi_{\mathrm{en}}(KHI(L)) \sim \Delta_L(t_1,\dots,t_n)\cdot\prod_{j=1}^{n}(t_j-1),$$

where $\sim$ *means the equality holds for elements in* $\mathbb{Z}[H]/\pm H$*.*

Remark 1.2.7. The analogous result of Corollary 1.2.6 has been proved for link Floer homology in Heegaard Floer theory by Ozsváth-Szabó [OS08a]. For instanton theory, the case of the single-variable Alexander polynomial for links in S^3 was again obtained in [Lim10, KM10a], while the case of the multivariable polynomial was unknown before.

For an element in a group ring $\mathbb{Z}[G]$

$$x = \sum_{g\in G} c_g\cdot g, \text{ for } c_g \in \mathbb{Z},$$

define

$$\|x\| = \sum_{g\in G}|c_g|.$$

This is still well-defined for an element in $\mathbb{Z}[G]/\pm G$. By the construction of Euler characteristics, we have

$$\dim_{\mathbb{C}} SHI(M,\gamma) \geq \left\|\chi_{\mathrm{en}}(SHI(M,\gamma))\right\| \geq \left\|\chi_{\mathrm{gr}}(SHI(M,\gamma))\right\|. \tag{1.2.5}$$

To provide an example that the second inequality in (1.2.5) is not always sharp, and hence χ_{en} contains more information than χ_{gr}, we consider the following example.

Example 1.2.8. Consider the 1-cusped hyperbolic manifold $M = m006$ in the *Snappy* program [CDMW21]. We have $H_1(M;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}_5 \cong \mathbb{Z}\langle t,r\rangle/(5r)$. Suppose γ consists of two parallel copies of the curve of slope $(1,0)$. Then we have

$$\tau(M,\gamma) = 1+r+t+rt+r^2t-r^3t-r^4t+rt^2+r^2t^2,$$

and

$$\tau(M,\gamma)|_{r=1} = 1+1+t+t+t-t-t+t^2+t^2 = 2+t+2t^2.$$

Hence we have

$$\|\chi_{\mathrm{en}}(SHI(M,\gamma))\| = \|\tau(M,\gamma)\| = 9 \text{ and } \|\chi(SHI(M,\gamma))\| = \|\tau(M,\gamma)|_{r=1}\| = 5.$$

For a $(1,1)$-knot $K \subset Y$, if the lower bound from the enhanced Euler characteristic in Corollary 1.2.4 coincides with the upper bound from Theorem 1.1.5, then we figure out the precise dimension of $KHI(Y,K)$. Other than L-space knots, this trick also applies to the following family of $(1,1)$-knots called **constrained knots**.

Let T^2 be the torus obtained by the quotient map $\mathbb{R}^2 \to T^2$ that identifies (x,y) with $(x+m,y+n)$ for $m,n \in \mathbb{Z}$. Suppose p,q are integers satisfying $p > 0$ and $\gcd(p,q) = 1$. Let α_0 and β_0 be two simple closed curves on T^2 obtained from two straight lines in $\mathbb{R}^2$ of slopes 0 and p/q. Then (T^2,α_0,β_0) is called the **standard diagram** of a lens space $L(p,q)$. Let $\alpha_1 = \alpha_0$ and let β_1 be a simple closed curve on T^2 such that it is disjoint from β_0 and $[\beta_1] = [\beta_0] \in H_1(T^2;\mathbb{Z})$. Then (T^2,α_1,β_1) is also a Heegaard diagram of $L(p,q)$. Let z and w be two basepoints in $T^2 - \alpha_0 \cup \beta_0 \cup \beta_1$.

The knot defined by the doubly-pointed diagram $(T^2,\alpha_1,\beta_1,z,w)$ is called a **constrained knot** and the diagram is called the **standard diagram** of the constrained knot. We will show that constrained knots are parameterized by five integers, which will be denoted by $C(p,q,l,u,v)$. For some technical reason, the knot $C(p,q,l,u,v)$ is in $L(p,q')$, where $qq' \equiv 1 \pmod p$. An example is shown in Figure 1.1, where (T^2,α_0,β_0) is the standard diagram of $L(5,2)$ and $(T^2,\alpha_1,\beta_1,z,w)$ defines $C(5,3,2,3,1)$.

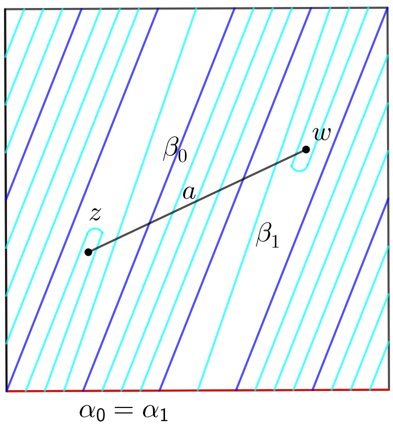


Figure 1.1 A constrained knot in $L(5,2)$.

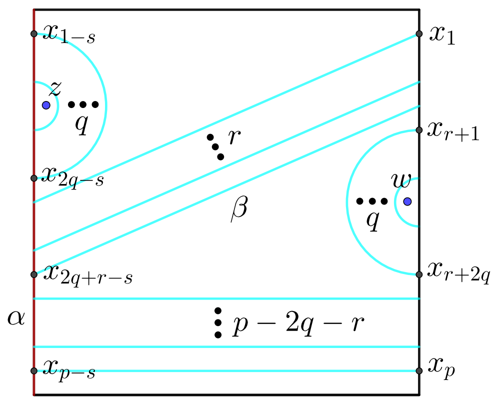


Figure 1.2 A $(1,1)$ diagram.

There is a complete classification of constrained knots [Ye21]. However, in this dissertation, we only point out that $\widehat{HFK}(Y,K)$ for a constrained knot is determined by the Turaev torsion of its complement. Indeed, Example 1.2.8 corresponds to the knot complement of a constrained knot. Then we have the following corollary.

Corollary 1.2.9. *Suppose Y is a lens space, and $K \subset Y$ is a $(1,1)$-knot. If K is either a (Heegaard Floer) L-space knot, or a constrained knot, then we have*

$$\dim_{\mathbb{C}} KHI(Y,K) = \dim_{\mathbb{F}_2} \widehat{HFK}(Y,K).$$

Proof. The case for an L-space knot follows from [RR17, Lemma 3.2], while the case for a constrained knot follows from the calculation of $\widehat{HFK}(Y,K)$ in Subsection 4.3.3. □

Remark 1.2.10. Greene-Lewallen-Vafaee [GLV18] provided a clear criterion to check if a $(1,1)$-knot is an L-space knot.

1.3 Calculation by Dehn surgery formulae

In this section, we describe the relation between $KHI(S^3,K)$ and $I^\sharp(S^3_n(K))$. The results in this section are based on [LY21c]. First, we propose the following definitions which are inspired by definitions in Remark 1.1.7.

Definition 1.3.1. A rational homology sphere Y is called an **instanton L-space** if $\dim_{\mathbb{C}} I^\sharp(Y) = |H_1(Y;\mathbb{Z})|$. A knot K in an instanton L-space Y is called an **instanton L-space knot** if a nontrivial surgery on it also gives an instanton L-space. We call K a **positive instanton L-space knot** if a positive surgery on it also gives an instanton L-space.

Remark 1.3.2. Note that Y is an instanton L-space if and only if $-Y$ is an instanton L-space. Since $S^3_r(\bar{K}) = -S^3_{-r}(K)$, a positive surgery on K gives an instanton L-space if and only if a negative surgery on the mirror knot $\bar{K}$ gives an instanton L-space.

Then we have the following theorem.

Theorem 1.3.3. *If $K \subset S^3$ is an instanton L-space knot, then K is a prime knot and there exists $k \in \mathbb{N}$ and integers*

$$n_k > n_{k-1} > \cdots > n_1 > n_0 = 0 > n_{-1} > \cdots > n_{1-k} > n_{-k} \text{ with } n_{-j} = -n_j$$

so that

$$\dim_{\mathbb{C}} KHI(S^3,K,S,i) = \begin{cases} 1 & \text{if } i = n_j \text{ for } j \in [-k,k], \\ 0 & \text{else,} \end{cases}$$

where the $\mathbb{Z}_2$-gradings of the generators of $KHI(S^3, K, S, n_j) \cong \mathbb{C}$ are alternating with respect to j.

Theorem 1.3.3 is an instanton analog of [OS05b, Theorem 1.2] in Heegaard Floer theory due to Ozsváth-Szabó. The key step to prove Theorem 1.3.3 is to establish an instanton version of the large surgery formula in Heegaard Floer theory. Before explaining more details about the proof, we state motivations and applications of the theorem. The construction of instanton Floer homology is related to flat $SU(2)$ connections, which correspond to homomorphisms from the fundamental group of the underlying manifold to $SU(2)$ (called $SU(2)$ representations). We propose the following definition.

Definition 1.3.4. An $SU(2)$ representation is called **abelian** if the image is contained in an abelian subgroup of $SU(2)$. An $SU(2)$ representation is called **irreducible** if it is not abelian. A knot $K \subset S^3$ is called **$SU(2)$-abundant** if the following two conditions hold.

(1) For all but finitely many $r \in \mathbb{Q}\backslash\{0\}$, the manifold $S^3_r(K)$ has an irreducible $SU(2)$ representation.

(2) For any $r = u/v \neq 0$ such that $S^3_r(K)$ has only abelian $SU(2)$ representations, there is some u-th root of unity ζ so that $\Delta_K(\zeta^2) = 0$.

Remark 1.3.5. The first condition implies K is not $SU(2)$**-averse** in the sense of [SZ20]. Note that if $b_1(Y) = 0$, then an $SU(2)$ representation of Y has abelian image if and only if it has cyclic image. The second condition corresponds to some nondegenerate condition in [BS18, Corollary 4.8]. By [BS19, Remark 1.6], when u is a prime power, $\Delta_K(\zeta^2) \neq 0$ for any K and any u-th root of unity ζ. Moreover, rationals with prime power numerators are dense in $\mathbb{Q}$.

Suppose $K \subset S^3$ is a nontrivial knot and $r \in \mathbb{Q}$. It is already known that if $|r| \leq 2$ [KM04a, Theorem 1] or $|r|$ is sufficiently large [SZ20, Corollary 1.2], then $S_r(K)$ has an irreducible $SU(2)$ representation. There are many other closed 3-manifolds with irreducible $SU(2)$ representations; see [KM04b, Lin16, Zen17, Zen18, BS18, LPCZ21, BS21b, SZ21, XZ21].

By [SZ20, Theorem 1.1] and [BS19, Corollary 4.8], if $K \subset S^3$ is not $SU(2)$-abundant, then K is an instanton L-space knot. Hence we obtain the following sufficient conditions for $SU(2)$-abundant knots by Theorem 1.3.3.

Theorem 1.3.6. *A nontrivial knot K is $SU(2)$-abundant unless all the following conditions hold.*

(1) There exists $k \in \mathbb{N}_+$ and integers $n_k > n_{k-1} > \cdots > n_1 > n_0 = 0$ so that

$$\pm\Delta_K(t) = (-1)^k + \sum_{j=1}^{k} (-1)^{k-j}(t^{n_j} + t^{-n_j}).$$

(2) The Seifert genus satisfies $g(K) = n_k = n_{k-1} + 1$.

(3) K is a prime knot, i.e., it is not a connected sum of two nontrivial knots.

Proof. If $K \subset S^3$ is not $SU(2)$-abundant, then K is an instanton L-space knot. By [BS18, Theorem 1.5] and passing to the mirror if necessary, we can further assume that for any sufficiently large integer n, the manifold $S^3_n(K)$ is an instanton L-space. Then Theorem 1.3.3 applies to K and we obtain Term (3). Since the space in the top $\mathbb{Z}$-grading of $KHI(S^3, K)$ is one-dimensional, it follows from [KM10b, Section 7] that K is fibred. Then by [BS22, Theorem 1.7], we know that $\dim_{\mathbb{C}} KHI(S^3, K, S, g(K) - 1) \geq 1$, and Theorem 1.3.3 forces equality to hold. Thus, Term (1) and Term (2) follow from

$$\sum_{i \in \mathbb{Z}} \chi(KHI(S^3, K, S, i)) \cdot t^i = \pm\Delta_K(t)$$

[Lim10, KM10a], where the sign ambiguity is due to the relative $\mathbb{Z}_2$-grading. □

Remark 1.3.7. By Term (1) and Term (2) in Theorem 1.3.6, we have

$$\det(K) = |\Delta_K(-1)| \leq 2k + 1 \leq 2g(K) + 1. \tag{1.3.1}$$

Remark 1.3.8. In [BS19, Theorem 1.5] and [BS21a, Corollary 1.7, and Proposition 5.4], Baldwin-Sivek proved that a nontrivial knot K is $SU(2)$-abundant unless K is both fibred and strongly quasi-positive (up to the mirror), the 4-ball genus $g_4(K)$ equals to $g(K)$, and the slope r with no irreducible $SU(2)$ representations satisfies $|r| \geq 2g(K) - 1$. It is worth mentioning that by techniques developed in this dissertation, it is possible to provide alternative proofs of those results.

From classification results in [OS05b, BM18, LV21], we have the following corollary.

Corollary 1.3.9. *The following knots are $SU(2)$-abundant.*

(1) Hyperbolic alternating knots, i.e., alternating knots that are not torus knots $T(2, 2n+1)$.

(2) Montesinos knots (including all pretzel knots), except torus knots $T(2, 2n+1)$, pretzel knots $P(-2, 3, 2n+1)$ for $n \in \mathbb{N}_+$ and their mirrors.

(3) Knots that are closures of 3-braids, except twisted torus knots $K(3,q;2,p)$ with $pq > 0$ and their mirrors, where $K(3,q;2,p)$ is the closure of a 3-braid made up of a $(3,q)$ torus braid with p full twist(s) on two adjacent strands.

Finally, we introduce a large surgery formula relating $KHI(S^3,K)$ and $I^\sharp(S^3_n(K))$. The constructions and results can be generalized to any rationally null-homologous knot K in a closed 3-manifold Y. For simplicity, we only discuss the constructions for a knot K in an integral homology sphere Y and deal with the general case in Chapter 5. Suppose S is a Seifert surface of K.

The large surgery formula in Heegaard Floer theory involves the filtered chain complex $CFK^-(Y,K)$. However, since there is no explicit construction of the chain complex of $KHI(Y,K)$, it is hard to construct the filtration directly in instanton theory. Fortunately, it is possible to construct some spectral sequence and then lift the spectral sequence to a filtered chain complex by an algebraic construction. Since we will use bypass maps based on contact geometry, it is more convenient to use manifolds with reverse orientations. We will construct two spectral sequences from $KHI(-Y,K)$ to $I^\sharp(-Y)$ by two types of bypass maps, and construct two filtered differentials d_+ and d_- on $KHI(-Y,K)$ with

$$H(KHI(-Y,K),d_+) \cong H(KHI(-Y,K),d_-) \cong I^\sharp(-Y).$$

Then we introduce the bent complex (*c.f.* Construction 5.1.25 and Construction 5.1.34) as follows. For any integer s, the **bent complex** and the **dual bent complex** are the chain complexes

$$A_s = A_s(-Y,K) := (KHI(-Y,K),d_s) \text{ and } A_s^\vee = A_s^\vee(-Y,K) := (KHI(-Y,K),d_s^\vee),$$

respectively, where for any element $x \in KHI(-Y,K,S,k)$,

$$d_s(x) = \begin{cases} d_+(x) & k > 0, \\ d_+(x)+d_-(x) & k = 0, \\ d_-(x) & k < 0, \end{cases} \text{ and } d_s^\vee(x) = \begin{cases} d_-(x) & k > 0, \\ d_+(x)+d_-(x) & k = 0, \\ d_+(x) & k < 0. \end{cases}$$

Since $d_+ \circ d_+ = d_- \circ d_- = 0$, we have $d_s \circ d_s = d_s^\vee \circ d_s^\vee = 0$. Hence we can consider the homologies $H(A_s)$ and $H(A_s^\vee)$.

Theorem 1.3.10 (Large surgery formula)**.** *Suppose K is a knot in an integral homology sphere Y. For a fixed integer n satisfying $|n| \geq 2g(K)+1$, suppose*

$$s_{min} = -|n| + 1 + g(K) \text{ and } s_{max} = |n| - 1 - g(K).$$

For any integer s', suppose $[s']$ is the image of s' in $\mathbb{Z}_{|n|}$. For any integer $s \in [s_{min}, s_{max}]$, we have

$$I^{\sharp}(-Y_{-n}(K), [s - s_{min}]) \cong \begin{cases} H(A_{-s}) & \text{if } n > 0, \\ H(A_{-s}^{\vee}) & \text{if } n < 0. \end{cases}$$

1.4 Extent of originality

Chapter 1 is introductory, where we state main results of this dissertation. Chapter 2 collects preliminaries on algebra and Floer homology from other people's work. Chapters 3, 4, 5, and the appendix are mostly based on the collaboration work of Zhenkun Li and the author (except Section 4.3, which is done solely by the author of this dissertation). Precisely, Chapter 3 is based on [LY22, Section 3.1-3.3], Section 4.1 is based on [LY21b, Section 4], Section 4.2 is based on [LY22, Section 4] and [LY21a, Section 3], Section 4.3 is based on [Ye21, Section 2-4], Chapter 5 is based on [LY21c, Section 3-5], and the appendix is based on [LY21b, Section 3] and [LY21a, Section 4].

Chapter 2

Preliminaries

In this chapter, we collect and restate some results that are known before except lemmas in Subsection 2.2.3 identifying mapping cones.

The first section contains conventions used in this dissertation. The second subsection is about algebra, especially homological algebra. This is one of main techniques used in the proofs of new results in this dissertation because Floer homologies can be regarded as graded vector spaces.

The second section is about (sutured) instanton Floer homology. We will omit some details and only explain carefully for topological constructions that needs to be unpackaged and used later.

2.1 Conventions

If it is not mentioned, all manifolds are smooth and oriented. Moreover, all manifolds are connected unless we indicate disconnected manifolds are also considered. For any compact 3-manifold M, we write $-M$ for the manifold obtained from M by reversing the orientation, called the **mirror manifold** of M. For any surface S in a compact 3-manifold M and any suture $\gamma \subset \partial M$, we write S and γ for the same surface and suture in $-M$, without reversing their orientations.

If it is not mentioned, homology groups and cohomology groups are with $\mathbb{Z}$ coefficients, *i.e.*, we write $H_*(Y)$ for $H_*(Y;\mathbb{Z})$. For other coefficients like $\mathbb{Q}$, we still write $H_*(M;\mathbb{Q})$. A general field is denoted by $\mathbb{F}$, and the field with two elements is denoted by $\mathbb{F}_2$. We write $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$.

A **rational homology sphere** is a closed 3-manifold whose homology groups with rational coefficients are isomorphic to those of S^3. An **integral homology sphere** is defined similarly. A knot $K \subset Y$ is called **null-homologous** if it represents the trivial homology class in $H_1(Y;\mathbb{Z})$, while it is called **rationally null-homologous** if it represents the trivial homology class in $H_1(Y;\mathbb{Q})$.

For a simple closed curve on a surface, we do not distinguish between its homology class and itself. The algebraic intersection number of two curves α and β on a surface is denoted by $\alpha\cdot\beta$, while the number of intersection points between α and β is denoted by $|\alpha\cap\beta|$. A basis (m,l) of $H_1(T^2;\mathbb{Z})$ satisfies $m\cdot l=-1$. The **surgery** means the Dehn surgery and the slope q/p in the basis (m,l) corresponds to the curve $qm+pl$.

For a manifold M, let $\operatorname{int}(M)$ denote its interior. For a submanifold A in a manifold Y, let $N(A)$ denote the tubular neighborhood. The knot complement of K in Y is denoted by $Y\backslash K:=Y\backslash\operatorname{int}(N(K))$. If we want to focus on the knot, we will also use $E(K)$ to denote the knot complement. Note that $\partial Y\backslash K\cong T^2$. We write $Y_r(K)$ for the manifold obtained from Y by a r-surgery (with respect to some given basis of $H_1(\partial Y\backslash K;\mathbb{Z})$).

For a knot K in a 3-manifold Y, we write $(-Y,K)$ for the induced knot in $-Y$ with induced orientation, called the **mirror knot** of K. The corresponding balanced sutured manifold is $(-Y\backslash K,-\gamma_K)$. In S^3, the mirror knot is also denoted by $\bar{K}$.

An argument holds for **large enough** n if there exists a fixed $N\in\mathbb{Z}$ so that the argument holds for any integer $n>N$. An argument holds for **small enough** n if there exists a fixed $N\in\mathbb{Z}$ so that the argument holds for any integer $n<N$.

2.2 Preliminaries on Algebra

2.2.1 Projectively transitive systems

In this subsection, we introduce the definition of the projectively transitive system. Note that Floer homology will be regarded as a projectively transitive system later.

Definition 2.2.1 ([JTZ21, BS15])**.** A **projectively transitive system** of vector spaces over a field $\mathbb{F}$ consists of

(1) a set A and collection of vector spaces $\{V_\alpha\}_{\alpha\in A}$ over $\mathbb{F}$,

(2) a collection of linear maps $\{g^\alpha_\beta\}_{\alpha,\beta\in A}$ well-defined up to multiplication by a unit in $\mathbb{F}$ such that

 (a) g^α_β is an isomorphism from V_α to V_β for any $\alpha,\beta\in A$, called a **canonical map**,

 (b) $g^\alpha_\alpha\doteq\mathrm{id}_{V_\alpha}$ for any $\alpha\in A$,

 (c) $g^\beta_\gamma\circ g^\alpha_\beta\doteq g^\alpha_\gamma$ for any $\alpha,\beta,\gamma\in A$, where $\doteq$ means the equation holds up to multiplication by a unit in $\mathbb{F}$. A morphism of projectively transitive systems of vector spaces over a field $\mathbb{F}$ from $(A, \{V_\alpha\}, \{g^\alpha_\beta\})$ to $(B, \{U_\gamma\}, \{h^\gamma_\delta\})$ is a collection of maps $\{f^\alpha_\gamma\}_{\alpha \in A, \gamma \in B}$ such that

(1) f^α_γ is a linear map from V_α to U_γ well-defined up to multiplication by a unit in $\mathbb{F}$ for any $\alpha \in A$ and $\gamma \in B$,

(2) $f^\beta_\delta \circ g^\alpha_\beta \doteq h^\gamma_\delta \circ f^\alpha_\gamma$ for any $\alpha, \beta \in A$ and $\gamma, \delta \in B$.

A **transitive system** of vector spaces over a field $\mathbb{F}$ is a projectively transitive system where equations with $\doteq$ are replaced by ones with the true equation $=$. A morphism of transitive systems of vector spaces over a field $\mathbb{F}$ is defined similarly.

We can replace vector spaces with groups or chain complexes of vector spaces and define the projectively transitive system and the transitive system similarly.

Remark 2.2.2. A transitive system of vector spaces $(A, \{V_\alpha\}, \{g^\alpha_\beta\})$ over a field $\mathbb{F}$ canonically defines an actual vector space over $\mathbb{F}$

$$V := \coprod_{\alpha \in A} V_\alpha / \sim,$$

where $v_\alpha \sim v_\beta$ if and only if $g^\alpha_\beta(v_\alpha) = v_\beta$ for any $v_\alpha \in V_\alpha$ and $v_\beta \in V_\beta$. A morphism of transitive systems of vector spaces canonically defines an linear map between corresponding actual vector spaces.

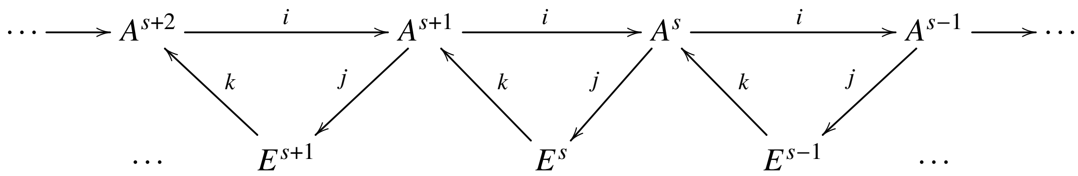
A projectively transitive system does not correspond to an actual vector space, but we can still choose representatives of vector spaces and maps, at the cost of introducing units in all equations of maps.

Convention. If $\mathbb{F} = \mathbb{F}_2$, the a projectively transitive system over $\mathbb{F}$ is simply a transitive system since $\mathbb{F}_2$ has only one unit. In this case, we do not distinguish the projectively transitive system, the transitive system, and the corresponding actual vector space. For a general field $\mathbb{F}$, we also do not distinguish the projectively transitive system and a representative of it, and add units for all equations.

2.2.2 Unrolled exact couples

In this subsection, we explain the construction of the spectral sequence from an unrolled exact couple [Boa99] and describe the relationship between the spectral sequence and the filtered chain complex.

Definition 2.2.3. An **unrolled exact couple** (E^s, A^s) is a diagram of graded vector spaces and homomorphisms of the form



in which each triangle

$$\cdots \to A^{s+1} \to A^s \to E^s \to A^{s+1} \to \cdots$$

is a long exact sequence. An unrolled exact couple is called **bounded** by an interval $[s_1, s_2]$ if $E^s = 0$ for $s \notin [s_1, s_2]$. A morphism between two unrolled exact couples (E^s, A^s) and $(\bar{E}^s, \bar{A}^s)$ consists of maps $f^s : E^s \to \bar{E}^s$ and $g^s : A^s \to \bar{A}^s$ that make all square commute.

Suppose (E^s, A^s) is an unrolled exact couple. For any integers s and r, define

$$\mathrm{Ker}^r A^s = \mathrm{Ker}(i^{(r)} : A^s \to A^{s-r}) \text{ and } \mathrm{Im}^r A^s = \mathrm{Im}(i^{(r)} : A^{s+r} \to A^s),$$

where $i^{(r)}$ denotes the r-fold iterate of i. There are subgroups of E^s:

$$0 = B_1^s \subset B_2^s \subset \cdots \subset \mathrm{Im}\, j = \mathrm{Ker}\, k \subset \cdots \subset Z_2^s \subset Z_1^s = E^s,$$

where

$$B_r^s = j(\mathrm{Ker}^{r-1} A^s) \text{ and } Z_r^s = k^{-1}(\mathrm{Im}^{r-1} A^{s+1}).$$

We call B_r^s and Z_r^s the r-th **boundary subgroup** and the r-th **cycle subgroup** of E^s, respectively. We call the quotient

$$E_r^s = Z_r^s / B_r^s$$

the s-component of the r-th **page**. Note that $E_1^s = E^s$. If the unrolled exact couple is bounded by $[s_1, s_2]$, then we call the direct sum

$$E_r = \bigoplus_{s_1}^{s_2} E_r^s$$

the r-th **page**.

Remark 2.2.4. If the unrolled exact couple (E^s, A^s) is bounded by $[s_1, s_2]$, then for any integers $r_1, r_2 > s_2 - s_1$ and any integer s, we have

$$B^s_{r_1} = B^s_{r_2}, Z^s_{r_1} = Z^s_{r_2}, E^s_{r_1} = E^s_{r_2} = E^s_\infty, \text{ and } E_{r_1} = E_{r_2} = E_\infty.$$

Proposition 2.2.5 ([Boa99, Section 0]). *Suppose (E^s, A^s) is an unrolled exact couple. For any integers s and r, there exists a well-defined map*

$$d^s_r : E^s_r \to E^{s+r}_r$$

induced by $j \circ (i^{(r-1)})^{-1} \circ k$ such that

$$d^{s+r}_r \circ d^s_r = 0 \text{ and } \operatorname{Ker} d^s_r / \operatorname{Im} d^{s-r}_r \cong E^s_{r+1}.$$

Equivalently, the set $\{(E^s_r, d^s_r)\}_{r\geq 1}$ forms a spectral sequence. Moreover, a morphism between two unrolled exact couples induces a map between the corresponding spectral sequences.

Boardman studied the convergence of the spectral sequence in Proposition 2.2.5 carefully, while we only need the special case for bounded unrolled exact couples.

Theorem 2.2.6 ([Boa99, Theorem 6.1]). *Suppose (E^s, A^s) is an unrolled exact couple bounded by $[s_1, s_2]$. Then by exactness we have*

$$A^{s_1} \cong A^{s_1-1} \cong A^{s_1-2} \cong \cdots \text{ and } A^{s_2+1} \cong A^{s_2+2} \cong A^{s_2+3} \cong \cdots$$

Consider the spectral sequence $\{(E_r, d_r)\}_{r\geq 1}$ from Proposition 2.2.5, where we omit the superscript s to denote the direct sum of all s-components. Then we have the following results.

(1) If $A^{s_1} = 0$, then $\{(E_r, d_r)\}_{r\geq 1}$ converges to $G = A^{s_2+1}$ with filtration $F^s G = \operatorname{Ker}^{s_2+1-s} A^{s_2+1}$ and we have $F^s G / F^{s+1} G \cong E^s_\infty$.

(2) If $A^{s_2+1} = 0$, then $\{(E_r, d_r)\}_{r\geq 1}$ converges to $G = A^{s_1}$ with filtration $F^s G = \operatorname{Im}^{s-s_1} A^{s_1}$ and we have $F^s G / F^{s+1} G \cong E^s_\infty$.

It is well-known that a filtered chain complex can induce a spectral sequence. Conversely, we may construct a filtered chain complex from a spectral sequence. However, *a priori* we may lose information when passing a filtered chain complex to a spectral sequence, so the reverse procedure is not always canonical. When fixing an inner product on the first page or equivalently fixing a basis, we have the following canonical construction.

Construction 2.2.7. Suppose (E^s, A^s) is an unrolled exact couple bounded by $[s_1, s_2]$ and suppose $\{(E_r, d_r)\}_{r\geq 1}$ is the spectral sequence from Proposition 2.2.5. Fix an inner product on $E_1^s = E^s$ for all integers s. For simplicity, we omit the superscript s and consider the direct sum E of all E^s.

For any subgroup X of E, there is a canonical isomorphism $E/X \cong X^\perp$, where $X^\perp$ is the orthogonal complement of X under the fixed inner product. From Definition 2.2.3 and Remark 2.2.4, there are subgroups of E:

$$0 = B_1 \subset B_2 \subset \cdots B_{s_2-s_1+1} \subset Z_{s_2-s_1+1} \subset \cdots \subset Z_2 \subset Z_1 = E.$$

For $p = 1, \dots, s_2 - s_1$, define B'_r as the orthogonal complement of B_p in B_{p+1}, define Z'_p as the orthogonal complement of Z_{p+1} in Z_p, and define E'_∞ as the orthogonal complement of $B'_{s_2-s_1+1}$ in $Z'_{s_2-s_1+1}$. Then we have

$$\begin{aligned} E_r = Z_r/B_r &\cong \bigoplus_{p=r}^{s_2-s_1} (B'_p \oplus Z'_p) \oplus E'_\infty, \\ \operatorname{Ker} d_r = Z_{r+1}/B_r &\cong \bigoplus_{p=r+1}^{s_2-s_1} (B'_p \oplus Z'_p) \oplus E'_\infty \oplus B'_r, \\ \operatorname{Im} d_r = B_{r+1}/B_r &\cong B'_r \end{aligned}$$

Hence we can lift $d_r : E_r \to E_r$ to a map

$$d'_r = I \circ d_r \circ P : E \to E,$$

where P and I are the projection and the inclusion, respectively. The only nontrivial part of d'_r is from Z'_r to B'_r, so for any $r_1, r_2 \in \{1, \dots, s_2 - s_1\}$, we have $d'_{r_1} \circ d'_{r_2} = 0$. Hence the summation

$$d = \sum_{r=1}^{s_2-s_1} d'_r$$

is a differential on E, *i.e.* $d^2 = 0$. Moreover, we have

$$H(E, d) \cong E'_\infty \cong E_{s_2-s_1+1} \cong E_\infty.$$

It is straightforward to check that the filtration $F^s E = \bigoplus_{p\geq s} E^p$ on (E, d) induces the spectral sequence $\{(E_r, d_r)\}_{r\geq 1}$.

2.2.3 The octahedral axiom

It is well-known that the derived category of an abelian category is a triangulated category (for example, see [Wei94, Proposition 10.2.4]). In particular, the derived category of the category of vector spaces is triangulated. Graded vector spaces can be regarded as objects in the derived category with trivial differentials. Many results in this subsection come from properties of the derived category of $\mathbb{Z}_2$-graded spaces. Note that, for a $\mathbb{Z}_2$-graded space, there is no difference between the chain complex and the cochain complex. Hence by saying a **complex** we mean a $\mathbb{Z}_2$-graded (co)chain complex, though all results apply to $\mathbb{Z}$-graded cochain complexes verbatim.

For a complex C and an integer n, we write C^n for its grading n part (under the natural map $\mathbb{Z}\to\mathbb{Z}_2$). With this notation, we suppose the differential d_C on C sends C^n to C^{n+1}. For any integer k, we write $C\{k\}$ for the complex obtained from C by the grading shift $C\{k\}^n = C^{n+k}$. We write $H(C,d_C)$ or $H(C)$ for the homology of a complex C with differential d_C.

A **chain map** is a map between complexes that commute with differentials. For a chain map $f : C \to D$, we write $f\{n\} : C\{n\} \to D\{n\}$ for the induced chain map and still write $f : H(C) \to H(D)$ for the induced map on the homology. Two chain maps $f,g : C \to D$ are **chain homotopic** if there is a map $h : C^n \to D^{n-1}$ for any n so that $f - g = h\circ d_C + d_D \circ h$. Two chain complexes C and D are **chain homotopy equivalent** if there are chain maps $f : C \to D$ and $g : D \to C$ so that $f\circ g$ and $g\circ f$ are chain homotopic to identities, where f and g are called **chain homotopy equivalences**.

For a chain map $f : C \to D$, we write $\mathrm{Cone}(f)$ for the **mapping cone** of f, *i.e.*, the complex consisting of the space $D\oplus C\{1\}$ and the differential

$$d_{\mathrm{Cone}(f)} := \begin{bmatrix} d_D & -f \\ 0 & -d_C \end{bmatrix}.$$

Then there is a long exact sequence

$$\cdots \to H(C) \xrightarrow{f} H(D) \xrightarrow{i} H(\mathrm{Cone}(f)) \xrightarrow{p} H(C)\{1\} \to \cdots$$

where i sends $x\in D$ to $(x,0)$ and p sends $(x,y)\in D\oplus C\{1\}$ to $-y$. If differentials of C and D are trivial, then we know

$$H(\mathrm{Cone}(f)) \cong \mathrm{Ker}(f)\oplus \mathrm{Coker}(f). \tag{2.2.1}$$

Remark 2.2.8. Our definitions about mapping cones follow from [Wei94], which are different from those in [OS08b, OS11].

Note that a triangulated category satisfies the octahedral axiom (for example, see [Wei94, Proposition 10.2.4]).

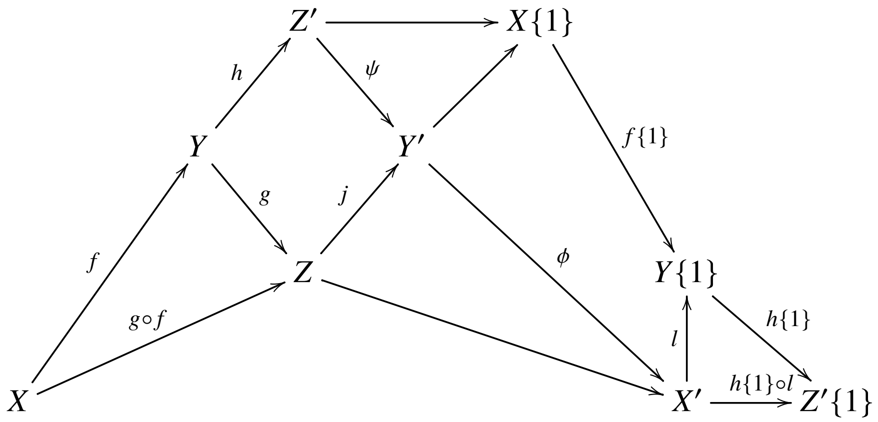
Lemma 2.2.9 (Octahedral axiom)**.** *Suppose X, Y, Z, X', Y', Z' are $\mathbb{Z}_2$-graded vector spaces satisfying the following long exact sequences*

$$\cdots \to X \xrightarrow{f} Y \xrightarrow{h} Z' \to X\{1\} \to \cdots$$
$$\cdots \to Y \xrightarrow{g} Z \to X' \xrightarrow{l} Y\{1\} \to \cdots$$
$$\cdots \to X \xrightarrow{g\circ f} Z \xrightarrow{j} Y' \to X\{1\} \to \cdots$$

Then we have the fourth long exact sequence

$$\cdots \to Z' \xrightarrow{\psi} Y' \xrightarrow{\phi} X' \xrightarrow{h\{1\}\circ l} Z'\{1\} \to \cdots$$

such that the following diagram commutes



(2.2.2)

where the arrows come from four long exact sequences.

Sketch of the proof. We regard graded vector spaces as chain complexes with trivial differentials. By the long exact sequences in the assumption, we know that Z', X', Y' are chain homotopic to mapping cones $\mathrm{Cone}(f), \mathrm{Cone}(g), \mathrm{Cone}(g \circ f)$, respectively. Define

$$\psi : Y \oplus X\{1\} \to Z \oplus X\{1\}$$
$$\psi(y,x) \mapsto (g(y),x)$$

and

$$\phi : Z \oplus X\{1\} \to Z \oplus Y\{1\}$$
$$\phi(z,x) \mapsto (z, f\{1\}(x))$$

The map ψ is a chain map from $\mathrm{Cone}(f)$ to $\mathrm{Cone}(g \circ f)$ and the map ϕ is a chain map from $\mathrm{Cone}(g \circ f)$ to $\mathrm{Cone}(g)$. Since the underlying vector space of $\mathrm{Cone}(\psi)$ is $Z \oplus X\{1\} \oplus Y\{1\} \oplus X\{2\}$, the inclusion $Z \oplus Y\{1\} \to Z \oplus X\{1\} \oplus Y\{1\} \oplus X\{2\}$ induces a map η from $\mathrm{Cone}(g)$ to $\mathrm{Cone}(\psi)$, which is a chain map and makes the following diagram commute

$$
\begin{array}{ccccccc}
\mathrm{Cone}(f) & \xrightarrow{\psi} & \mathrm{Cone}(g \circ f) & \xrightarrow{\phi} & \mathrm{Cone}(g) & \xrightarrow{h\{1\} \circ l} & \mathrm{Cone}(f)\{1\} \\
\downarrow{=} & & \downarrow{=} & & \downarrow{\eta} & & \downarrow{=} \\
\mathrm{Cone}(f) & \xrightarrow{\psi} & \mathrm{Cone}(g \circ f) & \longrightarrow & \mathrm{Cone}(\psi) & \longrightarrow & \mathrm{Cone}(f)\{1\}
\end{array}
$$

Define

$$
\begin{aligned}
\zeta : Z \oplus X\{1\} \oplus Y\{1\} \oplus X\{2\} &\to Z \oplus Y\{1\} \\
\zeta(z, x, y, x') &\mapsto (z, y + f\{1\}(x))
\end{aligned}
$$

Then we can check $\zeta \circ \eta$ is the identity map on $\mathrm{Cone}(g)$ and $\eta \circ \zeta$ is chain homotopic to the identity on $\mathrm{Cone}(\psi)$. Hence $\mathrm{Cone}(f), \mathrm{Cone}(g \circ f)$ and $\mathrm{Cone}(g)$ form a long exact sequence. □

Note that the chain homotopies in the proof of Lemma 2.2.9 are not canonical, and hence the maps ψ and ϕ are also not canonical. Thus, usually we cannot identify them with other given maps ψ', ϕ'. However, in the special case that $\phi \circ j = \phi' \circ j = 0$, it is possible to identify ϕ and ϕ' by the following lemma.

Lemma 2.2.10. *Suppose X, Y, Z, X', Y' are $\mathbb{Z}_2$-graded vector spaces satisfying the following horizontal exact sequences.*

$$
\begin{array}{ccccc}
Z & \xrightarrow{j} & Y' & \xrightarrow{l'} & X\{1\} \\
\downarrow{=} & & \phi \downarrow\downarrow \phi' & & \downarrow{f\{1\}} \\
Z & \xrightarrow{0} & X' & \xrightarrow{l} & Y\{1\}
\end{array}
$$

Suppose $\phi : Y' \to X'$ satisfies the two commutative diagrams, i.e., $\phi \circ j = 0$ and $f\{1\} \circ l' = l \circ \phi$. Suppose $\phi' : Y' \to X'$ satisfies the two commutative diagrams up to a unit, i.e., $\phi' \circ j = 0$ and $f\{1\} \circ l' = c \cdot l \circ \phi'$ for some $c \in \mathbb{C} \backslash \{0\}$. Then we have $\phi \doteq \phi'$ and hence $H(\mathrm{Cone}(\phi)) \cong H(\mathrm{Cone}(\phi'))$.

Proof. By exactness at X', we have

$$
\mathrm{Im}(\phi - c\phi') = \mathrm{Ker}(l) = \mathrm{Im}(0) = 0.
$$

Hence $\phi = c\phi'$. □

2.3 Preliminaries on instanton Floer homology

2.3.1 Instanton Floer homology for closed 3-manifolds

In this subsection, we review basic properties of instanton Floer homology for closed 3-manifolds.

Definition 2.3.1. Suppose Y is a closed 3-manifold and ω is a closed 1-submanifold in Y. Suppose that there is a closed oriented surface $\Sigma \subset Y$ of genus at least one such that the algebraic intersection number $\omega \cdot \Sigma$ is odd. Then the pair (Y,ω) is called an **admissible pair**.

For an admissible pair, Floer constructed a vector space by studying $SO(3)$ connections on Y and $Y \times \mathbb{R}$.

Theorem 2.3.2 ([Flo90])**.** *Suppose (Y,ω) is an admissible pair. Then there is a finite-dimensional complex vector space $I^{\omega}(Y)$ called the **instanton Floer homology** of (Y,ω).*

Suppose (Y_1,ω_1) and (Y_2,ω_2) are two admissible pairs. Suppose (W,ν) is a cobordism from (Y_1,ω_1) to (Y_2,ω_2), i.e. W is a 4-manifold with $\partial W = -Y_1 \sqcup Y_2$ and $\nu \subset W$ is a 2-submanifold with $\partial \nu = (-\omega_1) \sqcup \omega_2$.Then there exists a complex-linear map

$$I(W,\nu) : I^{\omega_1}(Y_1) \to I^{\omega_2}(Y_2),$$

*called the **cobordism map** associated to (W,ν).*

Remark 2.3.3. For a fixed 3-manifold Y, $I^{\omega}(Y)$ only depends on the class of ω in $H_1(Y;\mathbb{Z}_2)$.

For an admissible pair (Y,ω), any homology class $\alpha \in H_*(Y)$ induces a complex-linear action on the instanton Floer homology:

$$\mu(\alpha) : I^{\omega}(Y) \to I^{\omega}(Y).$$

For any two homology classes $\alpha_1,\alpha_2 \in H_*(Y)$, we have

$$\mu(\alpha_1+\alpha_2) = \mu(\alpha_1)+\mu(\alpha_2) \text{ and } \mu(\alpha_1)\mu(\alpha_2) = (-1)^{\deg(\alpha_1)\deg(\alpha_2)}\mu(\alpha_2)\mu(\alpha_1).$$

If $b_2(Y) > 0$, we can pick a basis $\beta_1,\ldots,\beta_n$ of $H_2(Y;\mathbb{Q})$ and consider the simultaneous generalized eigenspaces of all the actions $\mu(\beta_1),\ldots,\mu(\beta_n)$. The simultaneous eigenvalues, as a tuple $(\lambda_1,\ldots,\lambda_n)$, can be viewed as a linear map $\sum_{i=1}^{n} c_i\beta_i \mapsto \sum_{i=1}^{n} c_i\lambda_i$ from $H_2(Y;\mathbb{Q})$

to $\mathbb{Q}$ for coefficients $c_1, \ldots, c_n$. This linear map is the analog of the evaluation of the first Chern classes of spin^c structures in Heegaard Floer homology.

Moreover, there is a canonical $\mathbb{Z}_2$-grading on $I^\omega(Y)$ characterized by the following properties.

(1) The grading is compatible with the map $\mu(\alpha)$, *i.e.*, $\mu(\alpha)$ is homogeneous with respect to the grading.

(2) Suppose (W, ν) is a cobordism from (Y_1, ω_1) to (Y_2, ω_2). Then $I(W, \nu)$ is homogeneous with respect to this grading. Its degree can be calculated by the following formula

$$\deg(I(W,\nu)) \equiv \frac{1}{2}(\chi(W) + \sigma(W) + b_1(Y_2) - b_1(Y_1) + b_0(Y_2) - b_0(Y_1)) \pmod 2. \tag{2.3.1}$$

(3) Suppose Σ_g is a connected closed oriented surface of genus $g \geq 1$. Suppose $Y = S^1 \times \Sigma_g$ and $\Sigma = \{1\} \times \Sigma_g$. Then $I^{S^1}(Y|R)$ is supported in the odd grading.

Definition 2.3.4 ([KM10b, Definition 7.3])**.** Suppose (Y, ω) is an admissible pair, R is a closed surface of genus at least one, and $\omega \cdot R$ is odd. Let $I^\omega(Y|R)$ be the $(2g(R) - 2, 2)$-generalized eigenspaces of the pair of actions $(\mu(R), \mu(\mathrm{pt}))$ on $I^\omega(Y)$, where pt is any fixed basepoint on Y. It is $\mathbb{Z}_2$-graded.

Suppose M is a compact 3-manifold with torus boundary. Suppose $\omega \subset M$ is a closed 1-submanifold such that there exists a closed surface Σ of genus at least one with $\omega \cdot \Sigma$ odd. Let $i : \partial M \to M$ be the inclusion, and let

$$i_* : H_1(\partial M; \mathbb{Q}) \to H_1(M; \mathbb{Q}). \tag{2.3.2}$$

be the induced map on homology. Let $\gamma_1, \gamma_2, \gamma_3$ be three simple closed curves on ∂M with

$$\gamma_1 \cdot \gamma_2 = \gamma_2 \cdot \gamma_3 = \gamma_3 \cdot \gamma_1 = -1.$$

For $i \in \{1,2,3\}$, let Y_i be the closed 3-manifold obtained by Dehn filling along γ_i:

$$Y_i = M \bigcup_{\gamma_i = \{1\} \times \partial D^2} S^1 \times D^2.$$

Then clearly for $i \in \{1,2,3\}$, (Y_i, ω) are all admissible pairs. Floer proved the following theorem, usually referred to as the surgery exact triangle.

Theorem 2.3.5 ([Flo90])**.** *There is an exact triangle*

$$\begin{array}{ccc} I^{\omega}(Y_1) & \xrightarrow{\quad f_1 \quad} & I^{\omega}(Y_2) \\ {\scriptstyle f_3}\nwarrow & & \swarrow{\scriptstyle f_2} \\ & I^{\omega}(Y_3) & \end{array} \tag{2.3.3}$$

Furthermore, all maps in the exact triangle (2.3.3) are induced by cobordism maps.

Remark 2.3.6. In the original construction of Floer [Flo90] or Scaduto [Sca15, Section 2], one has to add some extra component to ω in one of Y_1, Y_2, and Y_3 to make the exact triangle hold. However, from [BS21a, Section 2.2], Baldwin-Sivek showed that one could wisely choose some other 1-submanifold ω' to start with. After adding the extra component coming from the original exact triangle, we finally arrive at a 1-submanifold representing the same homology class as ω in $H_1(Y;\mathbb{Z}_2)$ for all three 3-manifolds.

According to [KM07], whether the maps f_1, f_2, and f_3 in the above exact triangle are even or odd can be determined as follows.

Proposition 2.3.7 (Kronheimer and Mrowka [KM07, Section 42.3])**.** *In the exact triangle (2.3.3), we can determine the parities of the maps f_1, f_2, and f_3 as follows.*

(1) If there is an $i \in \{1,2,3\}$ so that $\gamma_i \cdot \delta = 0$, then f_{i-1} is odd and the other two are even. We take $f_0 = f_3$ in case $i = 1$. Recall δ is a nonzero element in $\ker(i_*)$ for the map i_* in Formula (2.3.2).

(2) If $\gamma_i \cdot \delta \neq 0$ for all $i \in \{1,2,3\}$, then there is a unique $j \in \{1,2,3\}$ so that $\gamma_j \cdot \delta$ and $\gamma_{j+1} \cdot \delta$ are of opposite signs. Note that we take $\gamma_4 = \gamma_1$ in case $j = 3$. Then the map f_j is odd and the other two are even.

With Proposition 2.3.7, the following lemma is straightforward.

Lemma 2.3.8. *In the exact triangle (2.3.3), after arbitrary simultaneous shifts on the canonical $\mathbb{Z}_2$ grading on $I^{\omega_i}(Y_i)$ for all $i \in \{1,2,3\}$, one and exactly one of the following two cases happens.*

(1) All three maps f_i are odd, and we have an equality

$$\chi(I^{\omega_1}(Y_1)) + \chi(I^{\omega_2}(Y_2)) + \chi(I^{\omega_3}(Y_3)) = 0.$$

(2) *There is an* $i \in \{1,2,3\}$ *so that* f_i *is odd and the other two are even, and we have an equality*

$$\chi(I^{\omega_{i-1}}(Y_{i-1})) = \chi(I^{\omega_i}(Y_i)) + \chi(I^{\omega_{i+1}}(Y_{i+1})).$$

Note here the indices are taken mod 3.

Remark 2.3.9. If there are no shifts, then clearly case (2) in Lemma 2.3.8 happens due to Proposition 2.3.7.

2.3.2 Sutured instanton Floer homology

In this subsection, we review basic properties of sutured instanton Floer homology for balanced sutured manifolds.

For a balanced sutured manifold (M,γ) (*c.f.* Definition 1.0.1), Kronheimer-Mrowka constructed sutured instanton Floer homology.

Theorem 2.3.10 ([KM10b, Section 7.4])**.** *For a balanced sutured manifold* (M,γ)*, one can associate a triple* (Y,R,ω)*, called a* ***closure*** *of* (M,γ)*, such that the following conditions hold.*

(1) Y *is a closed 3-manifold such that* M *is a submanifold of* Y.

(2) $R \subset Y$ *is a closed surface of genus at least one such that* $R_+(\gamma)$ *is a submanifold of* R *and* $R \cap \mathrm{int}(M) = \emptyset$.

(3) $\omega \subset Y$ *is a simple closed curve such that it intersects* R *transversely at one point and* $\omega \cap \mathrm{int}(M) = \emptyset$.

Moreover, the isomorphism class of $I^{\omega}(Y|R)$ *as in Definition 2.3.4 is independent of the choices of the triple* (Y,R,ω) *and is a topological invariant of* (M,γ).

Definition 2.3.11. For a balanced sutured manifold (M,γ), the vector space $I^{\omega}(Y|R)$ for a closure (Y,R,ω) of (M,γ) is called the **sutured instanton Floer homology** (or shortly sutured instanton homology) of (M,γ). It is also denoted by $SHI(M,\gamma)$ to stress the independence of choices of closures as claimed in Theorem 2.3.10.

The following are important properties of sutured instanton homology about tautness and productness.

Definition 2.3.12 ([Juh06, Definition 2.6])**.** A balanced sutured manifold (M,γ) is called **taut** if M is irreducible and $R(\gamma)$ is incompressible and Thurston norm-minimizing in $[R(\gamma)] \in H_2(M,\gamma)$.

Theorem 2.3.13 ([KM10b, Theorem 7.12] for SHI)**.** *Suppose (M,γ) is a balanced sutured manifold with M irreducible. Then (M,γ) is taut if and only if $SHI(M,\gamma)\neq 0$.*

Definition 2.3.14 ([Juh06, Juh08])**.** Suppose (M,γ) is a balanced sutured manifold. It is called a **homology product** if $H_1(M,R_+(\gamma))=0$ and $H_1(M,R_-(\gamma))=0$. It is called a **product sutured manifold** if

$$(M,\gamma)\cong([-1,1]\times\Sigma,\{0\}\times\partial\Sigma),$$

where Σ is a compact surface with boundary.

Theorem 2.3.15 ([KM10b, Theorem 7.18], based on [Ni07, Theorem 1.1])**.** *Suppose (M,γ) is a balanced sutured manifold and a homology product. Then (M,γ) is a product sutured manifold if and only if $SHI(M,\gamma)\cong\mathbb{C}$.*

In Theorem 2.3.10, only the isomorphism class of SHI is well-defined. Later, Baldwin-Sivek improved the naturality of SHI, making it possible to discuss elements in SHI. Similar work is done by Juhász-Thurston-Zemke [JTZ21] for SFH over $\mathbb{F}_2$, and Kutluhan-Sivek-Taubes [KST22] for sutured ECH.

Theorem 2.3.16 ([BS15, Section 9])**.** *For a balanced sutured manifold (M,γ) and any two closures (Y_1,R_1,ω_1) and (Y_2,R_2,ω_2) of (M,γ), there is an isomorphism*

$$\Phi_{1,2}:I^{\omega_1}(Y_1|R_1)\xrightarrow{\cong}I^{\omega_2}(Y_2|R_2),$$

which is well-defined up to multiplication by a unit in $\mathbb{C}$. Furthermore, the isomorphism Φ satisfies the following two conditions.

(1) If $(Y_1,R_1,\omega_1)=(Y_2,R_2,\omega_2)$, then

$$\Phi_{1,2}\doteq\mathrm{id},$$

where $\doteq$ means equal up to multiplication by a unit.

(2) If there is a third closure (Y_3,R_3,ω_3), then we have

$$\Phi_{1,3}\doteq\Phi_{2,3}\circ\Phi_{1,2}:I^{\omega_1}(Y_1|R_1)\to I^{\omega_3}(Y_3|R_3).$$

Moreover, these isomorphisms are homogeneous with respect to the canonical $\mathbb{Z}_2$-grading.

From Theorem 2.3.16, for a balanced sutured manifold (M,γ), Baldwin-Sivek [BS15, Section 9.2] constructed a projectively transitive system (*c.f.* Definition 2.2.1) based on the vector spaces $I^{\omega}(Y|R)$ coming from different closures of (M,γ) and the canonical maps Φ between them. This projectively transitive system is denoted by

$$\underline{\mathrm{SHI}}(M,\gamma),$$

which is the twisted refinement of SHI (there is another untwisted refinement **SHI** constructed in [BS15, Section 9.4] and used in Chapter 4). We can regard $\underline{\mathrm{SHI}}$ as a complex vector space well-defined up to multiplication by a unit, or an actual vector space at the cost of introducing units for equations of maps by Remark 2.2.2. From now on, we will write $\underline{\mathrm{SHI}}(M,\gamma)$ for the sutured instanton homology of (M,γ). Note that it has a relative $\mathbb{Z}_2$-grading from the canonical $\mathbb{Z}_2$ grading of instanton Floer homology. Hence we may consider its Euler characteristic up to sign.

For a knot and a closed 3-manifold, we have the following special case of sutured instanton homology.

Definition 2.3.17 ([KM10b, Section 7.6])**.** Suppose Y is a closed 3-manifold and $K\subset Y$ is a knot. Let $Y(1)$ be obtained from Y by removing a 3-ball and let δ be a simple closed curve on $\partial Y(1)$. Let γ_K consist of two meridians of K with opposite orientations. The **framed instanton homology** of Y is defined by

$$I^{\sharp}(Y):=\underline{\mathrm{SHI}}(Y(1),\delta),$$

which is isomorphic to $I^{S^1}(Y\sharp(S^1\times T^2)|\{1\}\times T^2)$ (*c.f.* [KM10b, Section 7.4]). The **instanton knot homology** of (Y,K) is defined by

$$\underline{\mathrm{KHI}}(Y,K):=\underline{\mathrm{SHI}}(Y\backslash K,\gamma_K),$$

which is a refinement of KHI.

Remark 2.3.18. In [BS15], in order to make the definition of $\underline{\mathrm{KHI}}$ independent of different choices of knot complements and the position of the meridional suture, Baldwin-Sivek also added a basepoint to the data. Also, the definition of $\underline{\mathrm{SHI}}(Y(1),\delta)$ depends on a choice of basepoint. We omit the basepoint from both notations.

2.3.3 Gradings associated to admissible surfaces

Suppose (M,γ) is a balanced sutured manifold and $S\subset M$ is a properly embedded surface. We state results by Li [Li19] and Ghosh-Li [GL19] about the decomposition of $\underline{\mathrm{SHI}}(M,\gamma)$ associated to S.

Definition 2.3.19 ([GL19, Definition 2.25])**.** Suppose (M,γ) is a balanced sutured manifold and $S \subset (M,\gamma)$ is a properly embedded surface in M. The surface S is called an **admissible surface** if the following conditions hold.

(1) Every boundary component of S intersects γ transversely and nontrivially.

(2) We require that $\frac{1}{2}|S\cap\gamma| - \chi(S)$ is an even integer.

For an admissible surface $S \subset (M,\gamma)$, there is a well-defined $\mathbb{Z}$ grading on $\underline{\mathrm{SHI}}(M,\gamma)$.

Theorem 2.3.20 ([Li19])**.** *Suppose (M,γ) is a balanced sutured manifold and $S \subset (M,\gamma)$ is an admissible surface with $n = \frac{1}{2}|S\cap\gamma|$. Then there exists a closure (Y,R,ω) of (M,γ) so that S extends to a closed surface $\bar{S} \subset Y$ with $\chi(\bar{S}) = \chi(S) - n$. Let $SHI(M,\gamma,S,i)$ denote the $(2i)$-generalized eigenspace of $\mu(\bar{S})$ acting on $SHI(M,\gamma) = I^{\omega}(Y|R)$. Then $SHI(M,\gamma,S,i)$ is preserved by the canonical maps in Theorem 2.3.16. Thus, we have the following decomposition*

$$\underline{\mathrm{SHI}}(M,\gamma) = \bigoplus_{i\in\mathbb{Z}} \underline{\mathrm{SHI}}(M,\gamma,S,i).$$

Furthermore, the following properties hold.

(1) If $|i| > \frac{1}{2}(n-\chi(S))$, then $\underline{\mathrm{SHI}}(M,\gamma,S,i) = 0$.

(2) If there is a sutured manifold decomposition $(M,\gamma) \overset{S}{\rightsquigarrow} (M',\gamma')$ (c.f. [Gab83, Section 3] and [Juh08, Definition 2.7]), then we have

$$\underline{\mathrm{SHI}}(M,\gamma,S,\frac{1}{2}(n-\chi(S))) \cong \underline{\mathrm{SHI}}(M',\gamma').$$

(3) For any $i \in \mathbb{Z}$, we have

$$\underline{\mathrm{SHI}}(M,\gamma,S,i) = \underline{\mathrm{SHI}}(M,\gamma,-S,-i).$$

(4) For any $i \in \mathbb{Z}$, we have

$$\underline{\mathrm{SHI}}(M,-\gamma,S,i) \cong \underline{\mathrm{SHI}}(M,\gamma,S,-i).$$

(5) For any $i \in \mathbb{Z}$, we have

$$\underline{\mathrm{SHI}}(-M,\gamma,S,i) \cong \mathrm{Hom}_{\mathbb{C}}(\underline{\mathrm{SHI}}(M,\gamma,S,-i),\mathbb{C}).$$

Remark 2.3.21. In [Li19], the grading was only constructed for an admissible surface with a connected boundary. When generalizing it to admissible surfaces with multiple boundary components, more choices arise in the construction of the grading. This new ambiguity was reduced to a combinatorial problem as discussed in [Li19, Section 3.3] and was then resolved in [Kav19].

Remark 2.3.22. Term (1) of Theorem 2.3.20 comes from the adjunction inequality of instanton Floer homology (*c.f.* [KM10b, Proposition 7.5]). Term (2) of Theorem 2.3.20 is a restatement of [KM10b, Proposition 7.11]. Term (3) is straighforward from the construction. Term (4) is from the isomorphism $I^{\omega}(Y|R) \cong I^{\omega}(Y|-R)$. Term (5) is from the pairing (*c.f.* [Li18]):

$$\langle\cdot,\cdot\rangle : \underline{\mathrm{SHI}}(M,\gamma)\times\underline{\mathrm{SHI}}(-M,\gamma)\to\mathbb{C}.$$

Suppose (M,γ) is a balanced sutured manifold, and $S\subset M$ is a properly embedded surface. If S is not admissible, then we isotop S to make it admissible.

Definition 2.3.23. Suppose (M,γ) is a balanced sutured manifold, and S is a properly embedded surface. A **stabilization** of S is a surface S' obtained from S by isotopy in the following sense. This isotopy creates a new pair of intersection points:

$$\partial S'\cap\gamma=(\partial S\cap\gamma)\cup\{p_+,p_-\}.$$

We require that there are arcs $\alpha\subset\partial S'$ and $\beta\subset\gamma$, oriented in the same way as $\partial S'$ and γ, respectively, and the followings hold.

(1) $\partial\alpha=\partial\beta=\{p_+,p_-\}$.

(2) α and β cobound a disk D with $\mathrm{int}(D)\cap(\gamma\cup\partial S')=\emptyset$.

The stabilization is called **negative** if ∂D is the union of arcs of α and β as an oriented curve. It is called **positive** if $\partial D=(-\alpha)\cup\beta$. See Figure 2.1. We denote by $S^{\pm k}$ the surface obtained from S by performing k positive or negative stabilizations, respectively.

The following lemma is straightforward.

Lemma 2.3.24. *Suppose (M,γ) is a balanced sutured manifold, and S is a properly embedded surface. Suppose S^+ and S^- are obtained from S by performing a positive and a negative stabilization, respectively. Then we have the following.*

(1) If we decompose (M,γ) along S or S^+ (c.f. [Gab83, Section 3] and [Juh08, Definition 2.7]), then the resulting two balanced sutured manifolds are diffeomorphic.

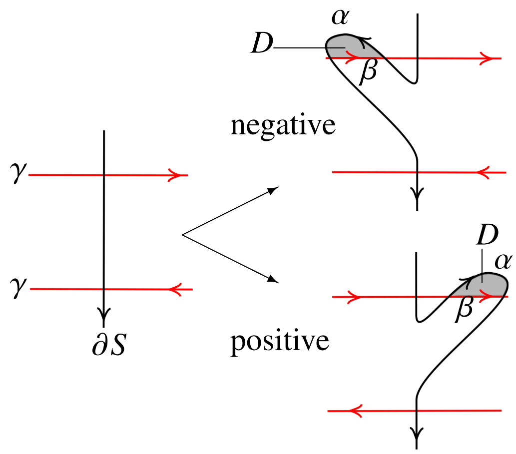


Figure 2.1 The positive and negative stabilizations of S.

(2) If we decompose (M,γ) along S^-, then the resulting balanced sutured manifold (M',γ') is not taut, as $R_\pm(\gamma')$ both become compressible.

Remark 2.3.25. The definition of stabilizations of a surface depends on the orientations of the suture and the surface. If we reverse the orientation of the suture or the surface, then positive and negative stabilizations switch between each other.

The following theorem relates the gradings associated to different stabilizations of the same surface.

Theorem 2.3.26 ([Li19, Proposition 4.3] and [Wan20, Proposition 4.17])**.** *Suppose (M,γ) is a balanced sutured manifold and S is a properly embedded surface in M that intersects γ transversely. Suppose all the stabilizations mentioned below are performed on a distinguished boundary component of S. Then, for any $p,k,l \in \mathbb{Z}$ such that the stabilized surfaces S^p and S^{p+2k} are both admissible, we have*

$$\underline{\mathrm{SHI}}(M,\gamma,S^p,l) = \underline{\mathrm{SHI}}(M,\gamma,S^{p+2k},l+k).$$

Note that S^p is a stabilization of S as introduced in Definition 2.3.23, and, in particular, $S^0 = S$.

Remark 2.3.27. The original form of Theorem 2.3.26 in [Li19] was stated for a Seifert surface in the case of a knot complement. However, it is straightforward to generalize the proof to the case of a general admissible surface in a general balanced sutured manifold, given the condition that the decompositions along S and $-S$ are both taut. This extra condition on taut decompositions was then dropped due to the work in [Wan20].

Convention. If $S \subset (M,\gamma)$ satisfies the conditions in Definition 2.3.19 except Term (2), then $\frac{1}{2}|S \cap \gamma| - \chi(S)$ is an odd integer. After a positive or negative stabilization, the surface S becomes admissible and induces a $\mathbb{Z}$-grading. By the grading shift behavior in Theorem 2.3.26, we may shift the $\mathbb{Z}$-grading by a half and consider the $(\mathbb{Z}+\frac{1}{2})$-grading associated to S. From now on, we consider either the $\mathbb{Z}$-grading or the $(\mathbb{Z}+\frac{1}{2})$-grading associated to a surface that might not be admissible.

If we have multiple admissible surfaces, then they together induce a multi-grading.

Theorem 2.3.28 ([GL19, Proposition 1.14]). *Suppose (M,γ) is a balanced sutured manifold and $S_1,\dots,S_n$ are admissible surfaces in (M,γ). Then there exists a $\mathbb{Z}^n$-grading on $\underline{\mathrm{SHI}}(M,\gamma)$ induced by $S_1,\dots,S_n$, which we write as*

$$\underline{\mathrm{SHI}}(M,\gamma) = \bigoplus_{(i_1,\dots,i_n)\in\mathbb{Z}^n} \underline{\mathrm{SHI}}(M,\gamma,(S_1,\dots,S_n),(i_1,\dots,i_n)).$$

Theorem 2.3.29 ([GL19, Theorem 1.12]). *Suppose (M,γ) is a balanced sutured manifold and $\alpha \in H_2(M,\partial M)$ is a nontrivial homology class. Suppose S_1 and S_2 are two admissible surfaces in (M,γ) such that*

$$[S_1] = [S_2] = \alpha \in H_2(M,\partial M).$$

Then, there exists a constant C so that

$$\underline{\mathrm{SHI}}(M,\gamma,S_1,l) = \underline{\mathrm{SHI}}(M,\gamma,S_2,l+C).$$

Based on the $\mathbb{Z}^n$ grading from Theorem 2.3.28, we can define the graded Euler characteristic.

Definition 2.3.30. Suppose (M,γ) is a balanced sutured manifold and $S_1,\dots,S_n$ are admissible surfaces in (M,γ) such that $[S_1],\dots,[S_n]$ generate $H_2(M,\partial M)$. Let $\rho_1,\dots,\rho_n \in H' = H_1(M)/\mathrm{Tors}$ satisfying $\rho_i \cdot S_j = \delta_{i,j}$. The **graded Euler characteristic** of $\underline{\mathrm{SHI}}(M,\gamma)$ is

$$\chi_{\mathrm{gr}}(\underline{\mathrm{SHI}}(M,\gamma)) := \sum_{(i_1,\dots,i_n)\in\mathbb{Z}^n} \chi(\underline{\mathrm{SHI}}(M,\gamma,(S_1,\dots,S_n),(i_1,\dots,i_n)))\cdot(\rho_1^{i_1}\cdots\rho_n^{i_n}) \in \mathbb{Z}[H']/\pm H'.$$

Remark 2.3.31. By Theorem 2.3.29, the definition of graded Euler characteristic is independent of the choices of $S_1,\dots,S_n$ if we regard it as an element in $\mathbb{Z}[H']/\pm H'$. If the admissible surfaces $S_1,\dots,S_n$ and a particular closure of (M,γ) are fixed, then the ambiguity of $\pm H'$ can be removed.

2.3.4 Contact handles and bypasses

Suppose $(M,\gamma) \subset (M',\gamma')$ is a proper inclusion of balanced sutured manifolds and suppose ξ is a contact structure on $M'\backslash \mathrm{int} M$ with dividing sets $\gamma' \cup (-\gamma)$. Baldwin-Sivek [BS16b] (see also [Li18]) constructed a **contact gluing map**

$$\Phi_\xi : \underline{\mathrm{SHI}}(-M,-\gamma) \to \underline{\mathrm{SHI}}(-M',-\gamma')$$

based on contact handle decompositions. Later, Li [Li18] showed that this map is functorial, *i.e.* it is independent of the contact handle decompositions and gluing two contact structures induces composite maps. In this subsection, we will describe the maps associated to contact 1- and 2-handle attachments, and bypass attachments (*c.f.* [Hon00]).

Contact 1-handle. Suppose D_- and D_+ are disjoint embedded disks in ∂M which each intersect γ in a single properly embedded arc. Consider the standard contact structure ξ_{std} on the 3-ball B^3. We glue $(D^2 \times [-1,1], \xi_{D^2}) \cong (B^3, \xi_{\mathrm{std}})$ to (M,γ) by diffeomorphisms

$$D^2 \times \{-1\} \to D_- \text{ and } D^2 \times \{+1\} \to D_+,$$

which preserve and reverse orientations, respectively, and identify the dividing sets with the sutures. Then we round corners as shown in Figure 2.2 (*c.f.* [BS16b, Figure 2]). Let (M_1,γ_1) be the resulting sutured manifold.

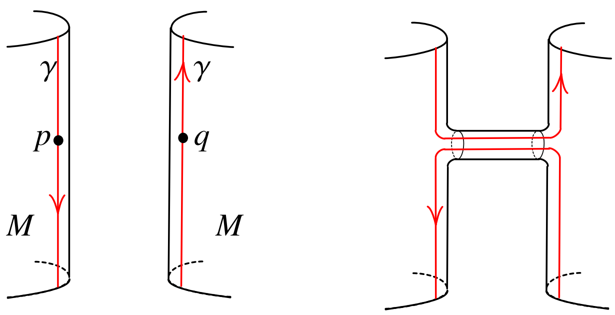


Figure 2.2 Left, the sutured manifold (M,γ) with two points p and q on the suture. Right, the 1-handle attachment along p and q.

Suppose (Y,R) is a closure of (M_1,γ_1). By [BS16b, Section 3.2], it is also a closure of (M,γ). Define the map associated to the contact 1-handle attachment by the identity map

$$C_{h^1} = C_{h^1,D_-,D_+} := \mathrm{id} : \underline{\mathrm{SHI}}(-M,-\gamma) \xrightarrow{=} \underline{\mathrm{SHI}}(-M_1,-\gamma_1).$$

Contact 2-handle. Suppose μ is an embedded curve in ∂M which intersects γ in two points. Let $A(\mu)$ be an annular neighborhood of λ intersecting γ in two cocores. We glue $(D^2\times[-1,1],\xi_{D^2})\cong(B^3,\xi_{\mathrm{std}})$ to (M,γ) by an orientation-reversing diffeomorphism

$$\partial D^2\times[-1,1]\to A(\mu),$$

which identifies positive regions with negative regions. Then we round corners as shown in Figure 2.3 (*c.f.* [BS16b, Figure 3]). Let (M_2,γ_2) be the resulting sutured manifold.

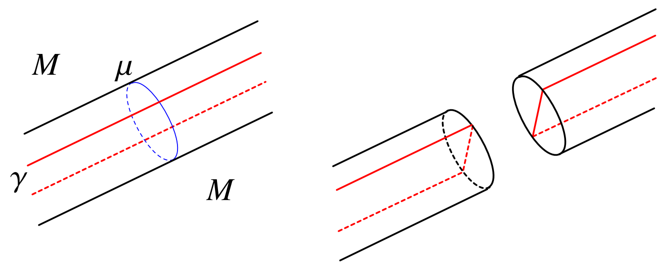


Figure 2.3 Left, the sutured manifold (M,γ) and the curve $\beta\subset\partial M$ that intersects γ at two points. Right, the 2-handle attachment along the curve μ.

We construct the map associated to the contact 2-handle attachment as follows. Let μ' be the knot obtained by pushing μ into M slightly. Suppose (N,γ_N) is the manifold obtained from (M,γ) by a 0-surgery along μ' with respect to the framing from ∂N. By [BS16b, Section 3.3], the sutured manifold (N,γ_N) can be obtained from (M_2,γ_2) by attaching a contact 1-handle. Since $\mu'\subset\mathrm{int}(M)$, the construction of the closure of (M,γ) does not affect μ'. Thus, we can construct a cobordism between closures of (M,γ) and (N,γ_N) by attaching a 4-dimensional 2-handle associated to the surgery on μ'. This cobordism induces a cobordism map

$$C_{\mu'}:\underline{\mathrm{SHI}}(-M,-\gamma)\to\underline{\mathrm{SHI}}(-N,-\gamma_N).$$

Consider the identity map

$$\iota:\underline{\mathrm{SHI}}(-M_2,-\gamma_2)\xrightarrow{=}\underline{\mathrm{SHI}}(-N,-\gamma_N).$$

Define the the map associated to the contact 2-handle attachment as

$$C_{h^2}=C_{h^2,\mu}:=\iota^{-1}\circ C_{\mu'}:\underline{\mathrm{SHI}}(-M,-\gamma)\to\underline{\mathrm{SHI}}(-M_2,-\gamma_2).$$

Bypass attachment. Suppose α is an embedded arc in ∂M which intersects γ in three points. Let D be a disk neighborhood of α intersecting γ in three arcs. There are six endpoints after cutting γ along α. We replace three arcs in D with another three arcs as shown in Figure 2.4. Let (M,γ') be the resulting sutured manifold. The arc α is called a **bypass arc** and this procedure is called **bypass attachment** along α.

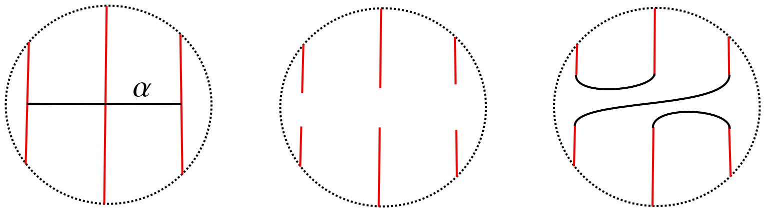


Figure 2.4 The bypass arc and the bypass attachment, where the orientation of ∂M is pointing out.

By Ozbagci [Ozb11, Section 3], the bypass attachment can be recovered by contact handle attachments as follows. First, one can attach a contact 1-handle along two endpoints of α. Then one can attach a contact 2-handle along a circle that is the union of α and an arc on the attached 1-handle. Topologically, the 1-handle and the 2-handle form a canceling pair, so the diffeomorphism type of the 3-manifold does not change. However, the contact structure is changed, and the suture γ is replaced by γ'. We define the **bypass map** associated to the bypass attachment as

$$\psi_\alpha := C_{h^2} \circ C_{h^1} : \underline{\mathrm{SHI}}(-M,-\gamma) \to \underline{\mathrm{SHI}}(-M,-\gamma').$$

We have some useful lemmas for bypass attachments.

Lemma 2.3.32. *Suppose (M,γ) is a balanced sutured manifold and $\alpha,\beta \subset \partial M$ are two bypass arcs with $\alpha \cap \beta = \emptyset$. Let ψ_α and ψ_β be the bypass maps associated to α and β, respectively. Let (M,γ') be the resulting balanced sutured manifold after bypass attachments along both α and β. Then we have*

$$\psi_\alpha \circ \psi_\beta \doteq \psi_\beta \circ \psi_\alpha : \underline{\mathrm{SHI}}(-M,-\gamma) \to \underline{\mathrm{SHI}}(-M,-\gamma').$$

Lemma 2.3.33. *Suppose (M,γ) is a balanced sutured manifold and $\alpha_0,\alpha_1 \subset \partial M$ are two bypass arcs. Suppose further that these two arcs are isotopic as bypass arcs, i.e., there is a smooth family α_t of bypass arcs for $t \in [0,1]$. Then α_1 and α_2 lead to isotopic balanced sutured manifold (M,γ'), and the bypass maps ψ_{α_1} and ψ_{α_2} are the same:*

$$\psi_{\alpha_1} = \psi_{\alpha_2} : \underline{\mathrm{SHI}}(-M,-\gamma) \to \underline{\mathrm{SHI}}(-M,-\gamma').$$

Remark 2.3.34. On the level of contact geometry, Honda has already proved Lemma 2.3.32 and Lemma 2.3.33 in [Hon00]. Thus, these two lemmas can also be proved by combining Honda's results with the functoriality of gluing maps Φ_ξ in [Li18].

Definition 2.3.35 ([Hon00, Section 3.4])**.** For a bypass arc α, let P_0, P_1, and P_2 be its three intersection points with γ, ordered by any orientation of α. For $i = 0,1,2$, let γ_i be the component of γ containing P_i. If $\gamma_0 = \gamma_1 \neq \gamma_2$ or $\gamma_1 = \gamma_2 \neq \gamma_0$, then α is called a **wave bypass**. If $\gamma_0 = \gamma_2 \neq \gamma_1$, then α is called an **anti-wave bypass**.

Remark 2.3.36. The names of wave and anti-wave follow from [GL16, Section 7], where waves and anti-waves are arcs whose endpoints are on the same curve. For an anti-wave bypass α, after removing the component of γ that only contains one intersection point, the arc α becomes a wave or an anti-wave.

Proposition 2.3.37 ([Hon02, Section 2.3])**.** *Suppose (M,γ) is a balanced sutured If α is a wave bypass, the suture γ_2 is obtained from γ_1 via a 'mystery move' (c.f. [Hon02, Figure 8]). If α is an anti-wave bypass, the suture γ_2 is obtained from γ_1 via a positive Dehn twist on ∂M. In both cases, the numbers of components of γ_1 and γ_2 are the same.*

Moreover, there is a bypass exact triangle for sutured instanton homology proved by Baldwin-Sivek.

Theorem 2.3.38 ([BS22, Theorem 1.20])**.** *Suppose (M,γ_1), (M,γ_2), (M,γ_3) are balanced sutured manifolds such that the underlying 3-manifolds are the same, and the sutures γ_1, γ_2, and γ_3 only differ in a disk shown in Figure 2.5. Then there exists an exact triangle*

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(-M,-\gamma_1) & \xrightarrow{\psi_1} & \underline{\mathrm{SHI}}(-M,-\gamma_2) \\ {\scriptstyle \psi_3}\nwarrow & & \swarrow{\scriptstyle \psi_2} \\ & \underline{\mathrm{SHI}}(-M,-\gamma_3) & \end{array} \tag{2.3.4}$$

Moreover, the maps ψ_i are induced by cobordisms, hence are homogeneous with respect to the relative $\mathbb{Z}_2$ grading on $\underline{\mathrm{SHI}}(M,\gamma_i)$.

The following proposition is straightforward from the description of the bypass map.

Proposition 2.3.39. *Suppose (M,γ) is a balanced sutured manifold and $S \subset (M,\gamma)$ is an admissible surface. Suppose the disk as in Figure 2.5, where we perform the bypass change, is disjoint from ∂S. Let γ_2 and γ_3 be the resulting two sutures. Then all the maps in the*

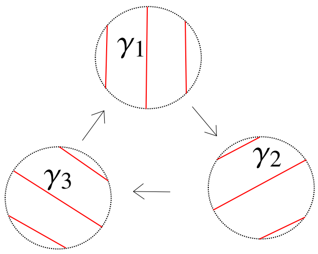


Figure 2.5 The bypass triangle.

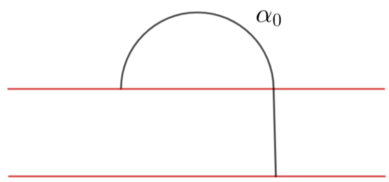


Figure 2.6 A trivial bypass.

bypass exact triangle (2.3.4) are grading preserving, i.e., for any $i \in \mathbb{Z}$, we have an exact triangle

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(-M,-\gamma_1,S,i) & \xrightarrow{\psi_{1,i}} & \underline{\mathrm{SHI}}(-M,-\gamma_2,S,i) \\ & {\scriptstyle \psi_{3,i}} \nwarrow \quad \swarrow {\scriptstyle \psi_{2,i}} & \\ & \underline{\mathrm{SHI}}(-M,-\gamma_3,S,i) & \end{array}$$

where $\psi_{k,i}$ are the restriction of ψ_k in (2.3.4).

A special bypass arc α_0 is depicted in Figure 2.6, where the bypass attachment along α is called a **trivial bypass** (*c.f.* [Hon02, Section 2.3]). Attaching a trivial bypass does not change the suture on ∂M and induces a product contact structure on $\partial M \times I$. The functoriality of the contact gluing maps indicates the following proposition.

Proposition 2.3.40. *A trivial bypass on (M,γ) induces an identity map on $\underline{\mathrm{SHI}}(M,\gamma)$.*

Chapter 3

Calculation by Heegaard diagrams

In this chapter, we obtain an upper bound for the dimension of sutured instanton homology from the Heegaard diagram of the balanced sutured manifold.

In the first section, we prove a dimension inequality (Proposition 1.1.3) for a rationally null-homologous tangle in a balanced sutured manifold. The essential arguments are based on the surgery exact triangle (Theorem 2.3.5) and the bypass exact triangle (Theorem 2.3.38).

In the second section, we construct a tangle from the Heegaard diagram and then apply the dimension inequality to prove Theorem 1.1.1 and Proposition 1.1.4. Then we prove the dimension inequality for $(1,1)$-knots (Theorem 1.1.5) by induction on the number of intersection points in the Heegaard digaram.

3.1 A dimension inequality for tangles

3.1.1 Basic setups

In this subsection, we introduce some basic notations for the proof of the main result.

Definition 3.1.1 ([XZ19, Definition 1.1])**.** Suppose (M,γ) is a balanced sutured manifold. A **tangle** $T\subset(M,\gamma)$ is a properly embedded 1-submanifold such that $T\cap A(\gamma)=\emptyset$. A tangle T is called **balanced** if

$$|T\cap R_+(\gamma)|=|T\cap R_-(\gamma)|.$$

A component a of T is called **vertical** if a is an arc from $R_+(\gamma)$ to $R_-(\gamma)$. A tangle T is called **vertical** if every component of T is vertical. Note that vertical tangles are balanced.

Suppose $T\subset(M,\gamma)$ is a vertical tangle, we construct a new balanced sutured manifold (M_T,γ_T), where $M_T=M\backslash\mathrm{int}N(T)$ and γ_T is the union of γ and one meridian for each component of T.

Theorem 3.1.2 ([XZ19]). *Suppose (M,γ) is a balanced sutured manifold and suppose $T\subset(M,\gamma)$ is a balanced tangle. Then there is a finite-dimensional complex vector space $SHI(M,\gamma,T)$, whose isomorphism class is a topological invariant of the triple (M,γ,T).*

In particular, for a vertical tangle $T\subset(M,\gamma)$, there is an isomorphism

$$SHI(M,\gamma,T)\cong SHI(M_T,\gamma_T).$$

The main result of this section is Proposition 1.1.3, which we restate as follows. Note that setting $T=\alpha$ recovers the original proposition.

Proposition 3.1.3. *Suppose (M,γ) is a balanced sutured manifold and T is a vertical tangle in (M,γ). Let α be a component of T and let $T'=T\backslash\alpha$. Suppose (M_T,γ_T) and $(M_{T'},\gamma_{T'})$ are defined as in Definition 3.1.1 for T and T'. If $[\alpha]=0\in H_1(M,\partial M;\mathbb{Q})$, then we have*

$$\dim_{\mathbb{C}}\underline{\mathrm{SHI}}(-M_{T'},-\gamma_{T'})\leq\dim_{\mathbb{C}}\underline{\mathrm{SHI}}(-M_T,-\gamma_T).$$

Suppose T has components $T_1,\dots,T_m$ and $\alpha=T_1$. Let γ_i be the meridian of T_i for $i=1,\dots,m$ and then

$$\gamma_T=\gamma\cup\gamma_1\cup\cdots\cup\gamma_m.$$

Since α is rationally null-homologous, there exists a surface S in M with ∂S consisting of arcs $\beta_1,\dots,\beta_k\subset\partial M$ and q copies of α for some integers k and q. Here q is the order of α, *i.e.* $q[\alpha]=0\in H_1(M,\partial M)$.

The surface S can be modified into a properly embedded surface S_T in M_T as follows. First, for q arcs in ∂S parallel to α, we isotop them to be on $\partial N(\alpha)$. Then $\beta_1,\dots,\beta_k$ can be regarded as arcs on ∂M_T. Second, we can isotop S to make it intersect $T_2,\dots,T_m$ transversely. Then removing disks in $N(T_i)\cap S$ for all $i=2,\dots,m$ induces a properly embedded surface S_T in M_T. Note that ∂S_T intersects γ_1 at q points, one for each arc parallel to α, and the part of ∂S_T on $\partial N(T_i)$ consists of circles parallel to γ_i for $i=2,\dots,m$.

Suppose p_+ and p_- are the endpoints of α on $R_+(\gamma)$ and $R_-(\gamma)$, respectively. Choose an arc $\zeta_+\subset R_+(\gamma)$ connecting p_+ and γ. The arc ζ_+ induces an arc on $R_+(\gamma_T)$ connecting γ_1 to γ such that the part on $\partial N(\alpha)$ is parallel to α. We still denote this arc by ζ_+ for simplicity. Similarly we can choose an arc $\zeta_-\subset R_-(\gamma_T)$ connecting γ_1 to γ.

Let Γ_0 be obtained from γ_T by band sum operations along ζ_+ and ζ_-. Then let Γ_n be obtained from Γ_0 by twisting along $(-\gamma_1)$ for n times. Moreover, let Γ_+ be the suture as depicted in Figure 3.1 and let $\Gamma_-=\gamma_T$.

Remark 3.1.4. The construction of ζ_+ and ζ_- here is a little different from the one in [LY22, Section 3.2], where we used β_1 to construct $\zeta_\pm$ and removed a trivial tangle from M_T to

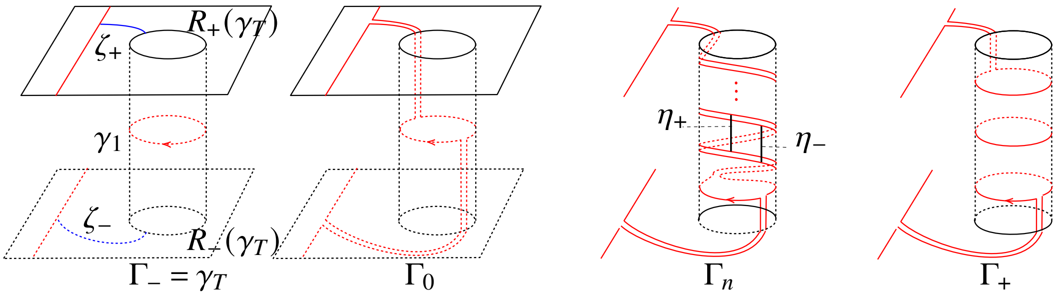


Figure 3.1 The arcs ζ_+, ζ_-, the sutures $\Gamma_-, \Gamma_0, \Gamma_n, \Gamma_+$, and the bypass arcs η_+, η_-.

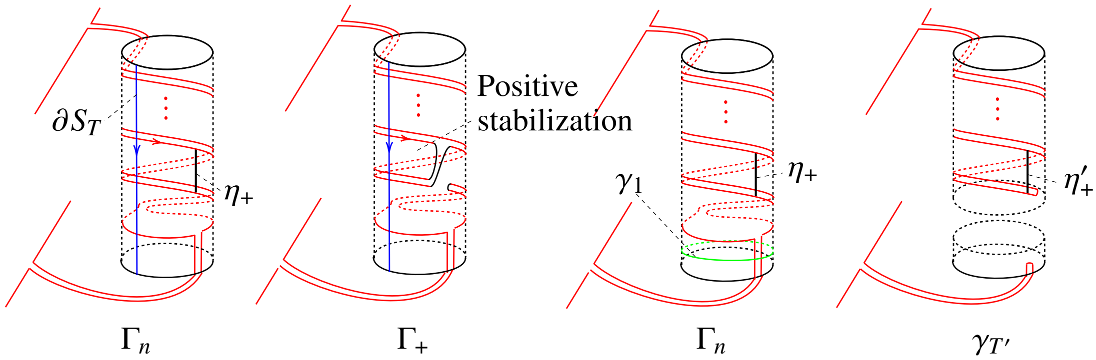


Figure 3.2 Left two subfigures, the bypass attachment along η_+. Right two subfigures, the bypass arcs before and after the contact 2-handle attachment.

obtain a manifold M_{T_0}. Hence the constructions of Γ_n and $\Gamma_\pm$ are also different. In particular, they were on M_{T_0} in the construction of [LY22, Section 3.2]. However, it turns out that removing the trivial tangle is not necessary and we can decompose M_{T_0} along a product disk to recover M_T in [LY22, Section 3.2, Step 3]. Thus, we can consider sutures on M_T and all results in [LY22, Section 3.2] apply without essential change. Also, the conditions that $\zeta_\pm$ are disjoint from $\beta_1, \ldots, \beta_k$ are not essential.

3.1.2 Graded bypass exact triangles

There are two straightforward choices of bypass arcs on Γ_n in the third subfigure of Figure 3.1, denoted by η_+ and η_-, respectively. It is straightforward to check that these two bypass arcs induce the following bypass exact triangles from Theorem 2.3.38 (*c.f.* the left two subfigures of Figure 3.2).

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(-M_T,-\Gamma_{n-1}) & \xrightarrow{\psi^{n-1}_{\pm,n}} & \underline{\mathrm{SHI}}(-M_T,-\Gamma_n) \\ & {\scriptstyle \psi^{\pm}_{\pm,n-1}} \nwarrow \quad \swarrow {\scriptstyle \psi^{n}_{\pm,\pm}} & \\ & \underline{\mathrm{SHI}}(-M_T,-\Gamma_{\pm}) & \end{array} \tag{3.1.1}$$

The bypasses are attached along η_+ and η_- from the exterior of the 3-manifold M_T, though the point of view in Figure 3.1 is from the interior of the manifold. So readers have to take extra care when performing these bypass attachments.

Since the bypass arcs η_+ and η_- are disjoint from ∂S_T, the bypass maps in the exact triangles (3.1.1) preserve gradings associated to S_T by Proposition 2.3.39. We describe it precisely as follows.

For any $j \in \mathbb{N} \cup \{-,+\}$, we write S_j for the surface S_T in $(-M,-\Gamma_j)$. Note that it induces a $\mathbb{Z}$-grading or a $(\mathbb{Z}+\frac{1}{2})$-grading. Then we define

$$i^j_{max} = |\frac{1}{2}(\frac{1}{2}|S_j \cap \Gamma_n| - \chi(S_j))| \text{ and } i^j_{min} = -|\frac{1}{2}(\frac{1}{2}|S_j \cap \Gamma_n| - \chi(S_j))|.$$

By Term (1) of Theorem 2.3.20, we know $\underline{\mathrm{SHI}}(-M,-\Gamma_j,S_j,i)$ vanishes when $i \notin [i^j_{min}, i^j_{max}]$. *A priori*, we do not know if $\underline{\mathrm{SHI}}$ is non-vanishing at the gradings i^j_{max} and i^j_{min}. Note that

$$\lim_{n\to+\infty} i^n_{max} = +\infty, \lim_{n\to+\infty} i^n_{min} = -\infty. \tag{3.1.2}$$

Definition 3.1.5. Suppose (M,γ) is a balanced sutured manifold and S is an admissible surface in (M,γ). For any $i,j \in \mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$, define

$$\underline{\mathrm{SHI}}(M,\gamma,S,i)[j] = \underline{\mathrm{SHI}}(M,\gamma,S,i-j).$$

Lemma 3.1.6. *For any $n \in \mathbb{N}$, we have two exact triangles, where all maps are grading preserving.*

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(-M_T,-\Gamma_n,S_n)[i^{n+1}_{min}-i^n_{min}] & \xrightarrow{\psi^{n}_{+,n+1}} & \underline{\mathrm{SHI}}(-M_T,-\Gamma_{n+1},S_{n+1}) \\ {\scriptstyle \psi^{+}_{+,n}} \uparrow & \swarrow {\scriptstyle \psi^{n+1}_{+,+}} & \\ \underline{\mathrm{SHI}}(-M_T,-\Gamma_+,S_+)[i^{n+1}_{max}-i^+_{max}] & & \end{array}$$

and

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(-M_T,-\Gamma_n,S_n)\,[i^{n+1}_{max}-i^{n}_{max}] & \xrightarrow{\psi^{n}_{-,n+1}} & \underline{\mathrm{SHI}}(-M_T,-\Gamma_{n+1},S_{n+1})\;. \\ \uparrow{\scriptstyle \psi^{-}_{-,n}} & \swarrow{\scriptstyle \psi^{n+1}_{-,-}} & \\ \underline{\mathrm{SHI}}(-M_T,-\Gamma_{-},S_{-})\,[i^{n+1}_{min}-i^{-}_{min}] & & \end{array}$$

Proof. This lemma follows directly from Proposition 2.3.39 and Theorem 2.3.26. □

From the vanishing results and the exact triangles in Lemma 3.1.6, the following lemma is straightforward. For any $i \in \mathbb{Z}, n \in \mathbb{N}$, let $\psi^{n,i}_{\pm,n+1}$ be the restriction of $\psi^{n}_{\pm,n+1}$ on the i-th grading associated to S_n.

Lemma 3.1.7. *The map*

$$\psi^{n,i}_{+,n+1} : \underline{\mathrm{SHI}}(-M_T,-\Gamma_n,S_n,i) \to \underline{\mathrm{SHI}}(-M_T,-\Gamma_{n+1},S_{n+1},i-(i^{n}_{min}-i^{n+1}_{min}))$$

is an isomorphism if

$$\begin{aligned} i < P_n &:= i^{n+1}_{max} + (i^{n}_{min} - i^{n+1}_{min}) - (i^{+}_{max} - i^{+}_{min}) \\ &= i^{n}_{min} + (i^{n+1}_{max} - i^{n+1}_{min}) - (i^{+}_{max} - i^{+}_{min}) \\ &= i^{n}_{min} + (n+1)q. \end{aligned}$$

Similarly, the map

$$\psi^{n,i}_{-,n+1} : \underline{\mathrm{SHI}}(-M_T,-\Gamma_n,S_n,i) \to \underline{\mathrm{SHI}}(-M_T,-\Gamma_{n+1},S_{n+1},i+(i^{n+1}_{max}-i^{n}_{max}))$$

is an isomorphism if

$$\begin{aligned} i > \rho_n &:= i^{n+1}_{min} - (i^{n+1}_{max} - i^{n}_{max}) + (i^{-}_{max} - i^{-}_{min}) \\ &= i^{n}_{max} - (i^{n+1}_{max} - i^{n+1}_{min}) + (i^{-}_{max} - i^{-}_{min}) \\ &= i^{n}_{max} - nq. \end{aligned}$$

3.1.3 An exact triangle from surgery

There is another important exact triangle induced by the surgery exact triangle.

Lemma 3.1.8. *For any $n \in \mathbb{N}$, there is an exact triangle*

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(-M_T,-\Gamma_n) & \xrightarrow{\qquad\qquad} & \underline{\mathrm{SHI}}(-M_T,-\Gamma_{n+1}) \\ & {\scriptstyle G_n}\nwarrow \qquad \swarrow{\scriptstyle F_{n+1}} & \\ & \underline{\mathrm{SHI}}(-M_{T'},-\gamma_{T'}) & \end{array} \tag{3.1.3}$$

Furthermore, we have commutative diagrams related to $\psi^n_{+,n+1}$ and $\psi^n_{-,n+1}$, respectively

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(-M_T,-\Gamma_n) & \xrightarrow{\psi^n_{\pm,n+1}} & \underline{\mathrm{SHI}}(-M_T,-\Gamma_{n+1}) \\ & {\scriptstyle G_n}\nwarrow \qquad \nearrow{\scriptstyle G_{n+1}} & \\ & \underline{\mathrm{SHI}}(-M_{T'},-\gamma_{T'}) & \end{array}$$

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(-M_T,-\Gamma_n) & \xrightarrow{\psi^n_{\pm,n+1}} & \underline{\mathrm{SHI}}(-M_T,-\Gamma_{n+1}) \\ & {\scriptstyle F_n}\searrow \qquad \swarrow{\scriptstyle F_{n+1}} & \\ & \underline{\mathrm{SHI}}(-M_{T'},-\gamma_{T'}) & \end{array}$$

Proof. Let γ_1' be the curve obtained by pushing γ_1 into the interior of M_T, with the framing from ∂M_T. Since γ_1' is in the interior of M_T, the surgeries do not influence the procedure of constructing closures of balanced sutured manifolds. Hence from Theorem 2.3.5 we have a $(0,1,\infty)$-surgery triangle associated to γ_1'.

$$\begin{array}{ccc} \underline{\mathrm{SHI}}((-M_T)_1,-\Gamma_{n+1}) & \xrightarrow{\qquad\qquad} & \underline{\mathrm{SHI}}((-M_T)_\infty,-\Gamma_{n+1}) \\ & \nwarrow \qquad \swarrow & \\ & \underline{\mathrm{SHI}}((-M_T)_0,-\Gamma_{n+1}) & \end{array}$$

The ∞-surgery does not change anything, so

$$((-M_T)_\infty,-\Gamma_{n+1}) \cong (-M_T,-\Gamma_{n+1}).$$

The 1-surgery is equivalent to a Dehn twist along γ_1'. It does not change the underlying 3-manifold, while the suture Γ_{n+1} is replaced by Γ_n:

$$((-M_T)_1,-\Gamma_{n+1}) \cong (-M_T,-\Gamma_n).$$

Finally, for the 0-surgery, from [BS16b, Section 3.3], we know that on the level of closures, performing a 0-surgery is equivalent to attaching a contact 2-handle along $\gamma_1 \subset \partial M_T$. Attaching such a contact 2-handle changes (M_T, Γ_{n+1}) to $(M_{T'}, \gamma_{T'})$. Hence we obtain the desired exact triangle.

We only prove the commutative diagram about G_n, G_{n+1} and $\psi^n_{+,n+1}$. The proofs for other diagrams are similar. First note that the curve γ_1' is disjoint from the bypass arc η_+. As a result, the related maps commute with each other by Lemma 2.3.32:

$$\psi^n_{+,n+1} \circ G_n \doteq G_{n+1} \circ \psi_{\eta_+'},$$

where η_+' is the bypass arc as shown in the last subfigure of Figure 3.2. It is straightforward to check that the bypass along η_+' is a trivial bypass, and by Proposition 2.3.40 it induces an identity map. Hence we conclude that

$$\psi^n_{+,n+1} \circ G_n \doteq G_{n+1} \circ \psi_{\eta_+'} \doteq G_{n+1} \circ \mathrm{id} = G_{n+1}.$$

□

Lemma 3.1.9. *For a large enough integer n, the map G_n in Lemma 3.1.8 is zero.*

Proof. We assume the lemma does not hold and derive a constradiction. For any n, there exists $x \in \underline{\mathrm{SHI}}(-M_{T'}, -\gamma_{T'})$ such that

$$y = G_n(x) \neq 0 \in \underline{\mathrm{SHI}}(-M_T, -\Gamma_n).$$

Suppose

$$y = \sum_{j \in \mathbb{Z}} y_j, \text{ where } y_j \in \underline{\mathrm{SHI}}(-M_T, -\Gamma_n, S_n, j),$$

$$j_{max} = \max_{y_j \neq 0} j \text{ and } j_{min} = \min_{y_j \neq 0} j.$$

By assumption j_{max} and j_{min} both exist and $j_{max} \geq j_{min}$. Suppose

$$z = G_{n+1}(x) \in \underline{\mathrm{SHI}}(-M_T, -\Gamma_{n+1}),$$

and similarly

$$z = \sum_{j \in \mathbb{Z}} z_j, \text{ where } z_j \in \underline{\mathrm{SHI}}(-M_T, -\Gamma_{n+1}, S_{n+1}, j).$$

From (3.1.2), we know that for a large enough integer n, we have

$$i^{n+1}_{max} + (i^n_{min} - i^{n+1}_{min}) - (i^+_{max} - i^+_{min}) > i^{n+1}_{min} - (i^{n+1}_{max} - i^n_{max}) + (i^-_{max} - i^-_{min}).$$

Hence at least one of the following two statements must be true.

(1) $j_{max} > i^{n+1}_{min} - (i^{n+1}_{max} - i^{n}_{max}) + (i^{-}_{max} - i^{-}_{min})$

(2) $j_{min} < i^{n+1}_{max} + (i^{n}_{min} - i^{n+1}_{min}) - (i^{+}_{max} - i^{+}_{min})$

We only work with the case where the first statement is true, and the other case is similar. From Lemma 3.1.8, we have

$$z = \psi^{n}_{+,n+1}(y) = \psi^{n}_{-,n+1}(y).$$

Suppose

$$i = j_{max} = j_{max} + (i^{n}_{max} - i^{n+1}_{max}),$$

and

$$j' = i + (i^{n}_{min} - i^{n+1}_{min}).$$

By Lemma 3.1.6, we have

$$\psi^{n,j'}_{+,n+1}(y_{j'}) = z_i = \psi^{n,j_{max}}_{-,n+1}(y_{j_{max}}).$$

Since $j' > j_{max}$, we have $z_i = 0$. By Lemma 3.1.7, the first statement implies $\psi^{n,j_{max}}_{-,n+1}$ is an isomorphism. Hence $y_{j_{max}} = 0$, which contradicts the assumption of j_{max}. □

Proof of Proposition 3.1.3. Suppose n is large enough. By the exact triangle (3.1.3), the fact that G_n is zero implies

$$\dim_{\mathbb{C}} \underline{\mathrm{SHI}}(-M_{T'}, -\gamma_{T'}) = \dim_{\mathbb{C}} \underline{\mathrm{SHI}}(-M_T, -\Gamma_{n+1}) - \dim_{\mathbb{C}} \underline{\mathrm{SHI}}(-M_T, -\Gamma_n).$$

From the exact triangle (3.1.1) and the fact that $\Gamma_- = \gamma_T$, we have

$$\dim_{\mathbb{C}} \underline{\mathrm{SHI}}(-M_T, -\gamma_T) \leq \dim_{\mathbb{C}} \underline{\mathrm{SHI}}(-M_T, -\Gamma_{n+1}) - \dim_{\mathbb{C}} \underline{\mathrm{SHI}}(-M_T, -\Gamma_n).$$

□

3.2 Heegaard diagrams and (1,1)-knots

3.2.1 Tangles from Heegaard diagrams

In this subsection, we introduce Heegaard diagrams of closed 3-manifolds and knots. We also give some constructions for tangles from Heegaard diagrams and then prove Theorem 1.1.1 and Proposition 1.1.4.

Definition 3.2.1. A **(genus** g**) diagram** is a triple (Σ,α,β), where

(1) Σ is a closed surface of genus g;

(2) $\alpha=\{\alpha_1,\dots,\alpha_m\}$ and $\beta=\{\beta_1,\dots,\beta_n\}$ are two sets of pair-wise disjoint simple closed curves on Σ. We do not distinguish the set and the union of curves.

Let N_0 be the manifold obtained from $\Sigma\times[-1,1]$ by attaching 3–dimensional 2–handles along $\alpha_i\times\{-1\}$ and $\beta_j\times\{1\}$ for each integer $i\in[1,m]$ and each integer $j\in[1,n]$. Let N be the manifold obtained from N_0 by capping off spherical boundaries. A diagram (Σ,α,β) is called **compatible** with a 3-manifold M if $M\cong N$. In such case, we also write M is compatible with (Σ,α,β), or (Σ,α,β) is a diagram of M.

Definition 3.2.2. A **(genus** g**) Heegaard diagram** is a (genus g) diagram (Σ,α,β) satisfying the following conditions.

(1) $|\alpha|=|\beta|=g$, *i.e.*, there are g curves in either tuple.

(2) $\Sigma\backslash\alpha$ and $\Sigma\backslash\beta$ are connected.

Given a Heegaard diagram (Σ,α,β), the manifolds compatible with $(\Sigma,\alpha,\emptyset)$ and $(\Sigma,\emptyset,\beta)$ are called the α**-handlebody** and the β**-handlebody**, respectively.

Definition 3.2.3. A **(genus** g**) doubly-pointed Heegaard diagram** $(\Sigma,\alpha,\beta,z,w)$ is a (genus g) Heegaard diagram with two points z and w in $\Sigma\backslash\alpha\cup\beta$. Let $a\subset\Sigma\backslash\alpha$ and $b\subset\Sigma\backslash\beta$ be two arcs connecting z to w. Suppose a' and b' are obtained from a and b by pushing them into α-handlebody and β-handlebody, respectively. A doubly-pointed Heegaard diagram $(\Sigma,\alpha,\beta,z,w)$ is called **compatible** with a knot K in a closed 3-manifold Y if (Σ,α,β) is compatible with Y and the union $a'\cup b'$ is isotopic to K.

Definition 3.2.4. Suppose (Σ,α,β) is a Heegaard diagram of a closed 3-manifold Y. A knot $K\subset Y$ is called the **core knot** of β_i for some $\beta_i\subset\beta$ if it is constructed as follows. Let M be the manifold compatible with the diagram $(\Sigma,\alpha,\beta\backslash\beta_i)$. It has a torus boundary and β_i induces a simple closed curve β_i' on ∂M. Dehn filling M along $\beta_i'\subset\partial M$ gives Y. Let K be the image of $S^1\times 0\subset S^1\times D^2$ under the filling map, where $S^1\times D^2$ is the filling solid torus.

The following is a basic fact in 3-dimensional topology.

Proposition 3.2.5 ([OS04b, Section 2.2]). *For any closed 3-manifold Y and any knot $K \subset Y$, there is a doubly-pointed Heegaard diagram compatible with (Y,K).*

In the rest of this subsection, we provide the construction of the balanced sutured handlebody (H,γ) used in Theorem 1.1.1.

Construction 3.2.6. Suppose Y is a closed 3-manifold and $K \subset Y$ is a knot. Suppose $(\Sigma,\alpha,\beta,z,w)$ is a genus $(g-1)$ doubly-pointed Heegaard diagram compatible with (Y,K). Consider the manifold M obtained from $\Sigma\times[-1,1]$ by attaching a 3-dimensional 1-handle along $\{z,w\}\times\{1\}$. Let Σ' be the component of ∂M with genus g. Let $\alpha_g \subset \Sigma'$ be the curve obtained by running from z to w and then back over the 1-handle. Let $\beta_g \subset \Sigma'$ be a small circle around z. Set

$$\alpha' = \alpha\times\{1\}\cup\{\alpha_g\} \text{ and } \beta' = \beta\times\{1\}\cup\{\beta_g\}.$$

Then (Σ',α',β') is a genus g Heegaard diagram compatible with Y. Since β_g is a meridian of K, the knot K is the core knot of β_g.

Construction 3.2.7. Suppose Y is a closed 3-manifold and $(\Sigma',\alpha',\beta' = \{\beta_1,\dots,\beta_g\})$ is a genus g Heegaard diagram compatible with Y. Let $Y(1)$ be obtained from Y by removing a 3-ball. The manifold $Y(1)$ can be obtained from the α'-handlebody by attaching 3–dimensional 2-handles along β_i for each integer $i \in [1,g]$. Note that a 3-dimensional 2-handle can be thought of as $[-1,1]\times D^2$ attached along $[-1,1]\times\partial D^2$. Let $\theta_i = [-1,1]\times\{0\}$ be the co-core of the 2-handle attached along β_i. We have a properly embedded tangle in $Y(1)$:

$$T = \theta_1\cup\cdots\cup\theta_g.$$

Pick a simple closed curve $\delta \subset \partial Y(1)$ such that for any i, two endpoints of θ_i lie on two different sides of δ. From the construction, the manifold $Y(1)_T = Y(1)\backslash N(T)$ is the α'-handlebody and the suture δ_T consists of all β_i curves and a curve β_{g+1} induced by δ, *i.e.*

$$\delta_T = \beta_1\cup\cdots\cup\beta_g\cup\beta_{g+1}.$$

Hence $R_+(\delta_T)$ and $R_-(\delta_T)$ can be obtained from $\Sigma\backslash\beta$ by cutting along β_g, which are both spheres with $(g+1)$ punctures.

Construction 3.2.8. Suppose Y is a closed 3-manifold and $K \subset Y$ is a knot. Suppose $(\Sigma,\alpha,\beta = \{\beta_1,\dots,\beta_{g-1}\},z,w)$ is a genus $(g-1)$ doubly-pointed Heegaard diagram of (Y,K).

We apply Construction 3.2.6 to obtain a genus g Heegaard diagram $(\Sigma',\alpha',\beta'=\{\beta_1,\dots,\beta_g\})$ of Y, and then apply Construction 3.2.7 to obtain a balanced sutured handlebody

$$(H,\gamma)=(Y(1)_T,\delta_T=\beta_1\cup\cdots\cup\beta_{g+1}).$$

Note that the diagram (Σ',α',β) is compatible with the knot complement $Y\backslash K$. Suppose β''_g and β''_{g+1} are curves on $\partial Y\backslash K$ induced by β_g and β_{g+1}, respectively. Since $\beta''_g\cap\beta''_{g+1}=\emptyset$ and $\partial Y\backslash K\cong T^2$, the curve β''_{g+1} is parallel to β''_g. Since β''_g is a meridian of K and $(Y\backslash K,\beta''_g\cup\beta''_{g+1})$ is a balanced sutured manifold, the curve β''_{g+1} must be another meridian of K with the orientation opposite to that of β''_g.

We provide an explicit construction of the curve $\beta_{g+1}\subset\partial H$ in Construction 3.2.8.

Construction 3.2.9. Suppose $(\Sigma',\alpha',\beta'=\{\beta_1,\dots,\beta_g\})$ is a genus g Heegaard diagram compatible with a closed 3-manifold Y. Let H be the α'-handlebody. For any integer $i\in[1,g]$, let β_i be oriented arbitrarily and let $\beta'_i\subset\partial H$ be the curve obtained by pushing off β_i to the right with respect to the orientation. Suppose β'_i is oriented reversely. Let β_{g+1} be the curve obtained from all of the β'_i by band sums with respect to orientations so that β_{g+1} is disjoint from $\beta_1,\dots,\beta_g$. Set

$$\gamma=\beta_1\cup\cdots\cup\beta_{g+1}.$$

It is straightforward to check that (H,γ) is the one obtained in Construction 3.2.8.

We can also obtain the original 3-manifold Y from the sutured handlebody (H,γ) as follows.

Construction 3.2.10. Suppose H is a handlebody, and γ is a suture on ∂H such that $R_+(\gamma)$ and $R_-(\gamma)$ are both spheres with $(g+1)$ punctures. Let $\Sigma=\partial H$. Suppose Σ has genus g. Let $\alpha_1,\dots,\alpha_g$ be boundaries of g compressing disks $D_1,\dots,D_g$ so that $H\backslash(D_1\cup\cdots\cup D_g)$ is a 3-ball. Since $R_+(\gamma)$ and $R_-(\gamma)$ are both spheres with $(g+1)$ punctures, the suture γ has $(g+1)$ components. We can take arbitrary g of them to form β. Then (Σ,α,β) is a Heegaard diagram. Let Y be a closed 3-manifold compatible with (Σ,α,β). Since different choices of such g curves from γ are related to each other by a finite sequence of handle slides, the manifold Y is well-defined up to diffeomorphism.

Let δ be the remaining component of γ and let T be the union of co-cores of β_i curves as in Construction 3.2.7. It is straightforward to check that $(Y(1)_T,\delta_T)=(H,\gamma)$.

Proof of Theorem 1.1.1. Suppose $(H,\gamma)=(Y(1)_T,\delta_T)$ and $(Y\backslash K,\beta''_g\cup\beta''_{g+1})$ are obtained from Construction 3.2.8. Note that $\beta''_g\cup\beta''_{g+1}$ are parallel copies of the meridian of K. Then we have

$$KHI(-Y,K)=\underline{\mathrm{SHI}}(-Y\backslash K,-(\beta''_g\cup\beta''_{g+1}))$$

by Definition 2.3.17. Since Y is a rational homology sphere, we have

$$H_1(Y(1), \partial Y(1); \mathbb{Q}) = 0.$$

In particular, any component of T has trivial rational homology class. Then the theorem follows from Proposition 1.1.3. □

Remark 3.2.11. Suppose (Σ, α, β) is a Heegaard diagram of a rational homology sphere Y and K is the core knot of β_i for some $\beta_i \subset \beta$. Suppose $(H, \gamma) = (Y(1)_T, \delta_T)$ is obtained from Construction 3.2.9. Then the proof of Theorem 1.1.1 applies without change, and we conclude the same inequality.

Proof of Proposition 1.1.4. Similar to the proof of Theorem 1.1.1, since the knot K has trivial rational homology class, the corresponding tangle has trivial homology class in $H_1(Y(1), \partial Y(1); \mathbb{Q})$. □

3.2.2 The instanton knot homology of (1,1)-knots

In this subsection, we use Theorem 1.1.1 to prove Theorem 1.1.5.

Definition 3.2.12. Suppose $p, q \in \mathbb{Z}$ satisfy $p \geq 1, 0 \leq q < p$ and $\gcd(p,q) = 1$. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be two straight lines in $\mathbb{R}^2$ passing the origin with slopes 0 and p/q, respectively, and let $r : \mathbb{R}^2 \to T^2$ be the quotient map induced by $(x,y) \to (x+m, y+n)$ for $m,n \in \mathbb{Z}$. Suppose $\alpha = r(\tilde{\alpha})$ and $\beta = r(\tilde{\beta})$. Then the manifold compatible with the Heegaard diagram (T^2, α, β) is called a **lens space** and is denoted by $L(p,q)$. Furthermore, the Heegaard diagram (T^2, α, β) is called the **standard diagram** of the lens space. In particular, we regard S^3 as a lens space $L(1,0)$.

The lens space is oriented so that the orientation on the α-handlebody is induced from the standard embedding of $S^1 \times D^2$ in $\mathbb{R}^3$. With this convention, the lens space $L(p,q)$ comes from the p/q-surgery on the unknot in S^3.

Definition 3.2.13. A proper embedded arc η in a handlebody H is called a **trivial arc** if there is an embedded disk $D \subset H$ satisfying $\partial D = \eta \cup (D \cap \partial H)$. The disk D is called the **cancelling disk** of η. A knot K in a closed 3-manifold Y admits a **(1,1)-decomposition** if the following conditions hold.

(1) Y admits a splitting $Y = H_1 \cup_{T^2} H_2$ so that $H_1 \cong H_2 \cong S^1 \times D^2$.

(2) $K \cap H_i$ is a properly embedded trivial arc in H_i for $i \in \{1, 2\}$.

In this case, Y is either a lens space or $S^1 \times S^2$. A knot K admitting a (1,1)-decomposition is called a $(1,1)$-**knot**.

Proposition 3.2.14 ([Ras05, Section 6.2] and [GMM05, Section 2]). *For $p,q,r,s \in \mathbb{N}$ satisfying $2q+r \le p$ and $s < p$, a (1,1)-decomposition of a knot determines and is determined by a doubly-pointed diagram. After isotopy, such a diagram becomes (T^2,α,β,z,w) in Figure 3.3, where p is the total number of intersection points, q is the number of strands around either basepoint, r is the number of strands in the middle band, and the i-th point on the right-hand side is identified with the $(i+s)$-th point on the left-hand side.*

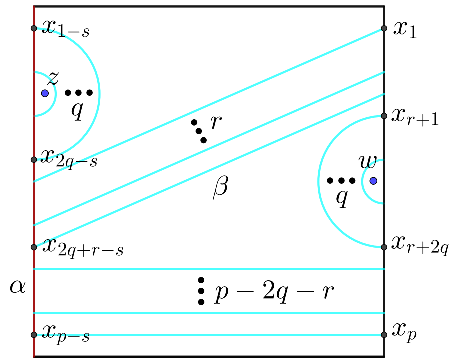


Figure 3.3 (1,1)-diagram.

Definition 3.2.15. A simple closed curve β on (T^2,α,z,w) is called **reduced** if the number of intersection points between α and β is minimal. The doubly-pointed diagram in Figure 3.3 is called the **(1,1)-diagram** of type (p,q,r,s), which is denoted by $W(p,q,r,s)$. Strands around basepoints are called **rainbows** and strands in the bands are called **stripes**.

If the **(1,1)-diagram** of $W(p,q,r,s)$ is a Heegaard diagram for some parameters (p,q,r,s), or equivalently, β has one component and represents a nontrivial homology class in $H_1(T^2)$, then the corresponding knot is also denoted by $W(p,q,r,s)$.

A $(1,1)$-knot whose (1,1)-diagram does not have rainbows is called a **simple knot** (*c.f.* [Ras07, Section 2.1]). For simple knots, let $K(p,q,k) = W(p,0,k,q)$.

Proposition 3.2.16. *The mirror knot of a $(1,1)$-knot $W(p,q,r,s)$ is*

$$W(p,q,p-2q-r,p-s+2q).$$

Proof. The Heegaard diagram of the mirror knot of $W(p,q,r,s)$ is obtained by the (1,1)-diagram of $W(p,q,r,s)$ by vertical reflection. We redraw the Heegaard diagram so that the lower band becomes the middle band and the middle band becomes the lower band. This proposition follows from the definition. □

According to [GMM05, Section 3] (also [OS04b, Section 6]), for the $\widehat{HFK}$ of a $(1,1)$-knot, the generators of the chain complexes are intersection points of α and β in the (1,1)-diagram and there is no differential. Thus, the following proposition holds.

Proposition 3.2.17. *For a* $(1,1)$*-knot* $K = W(p,q,r,s)$ *in* Y*, we have*

$$\mathrm{rk}_{\mathbb{Z}}\widehat{HFK}(Y,K) = \dim_{\mathbb{F}_2}\widehat{HFK}(Y,K) = p.$$

We restate Construction 3.2.9 more carefully.

Construction 3.2.18. Suppose (T^2,α,β,z,w) is the (1,1)-diagram of $W(p,q,r,s)$. We construct a sutured handlebody (H,γ) as follows, called the **(1,1)-sutured-handlebody** of $W(p,q,r,s)$.

(1) Let Σ be the genus-two boundary of the manifold obtained from $[-1,1]\times T^2$ by attaching a 3-dimensional 1-handle along $\{1\}\times\{z,w\}$. For simplicity, when drawing the diagram, the attached 1-handle will still be denoted by two basepoints z and w.

(2) Let α_1 and β_1 denote the curves on Σ induced from α and β, respectively. Let β be oriented so that the innermost rainbow around z is oriented clockwise, which induces an orientation of β_1. If there is no rainbow, let β be oriented so that each stripe goes from left to right in Figure 3.3.

(3) Consider the straight arc connecting z to w in Figure 3.3. It induces a simple closed curve α_2 on Σ by going along the 1-handle. Let β_2 be the curve on Σ induced by a small circle around z, oriented counterclockwise.

(4) Let γ_1 and γ_2 be obtained by pushing off β_1 and β_2 to the right with respect to the orientation. Suppose they are oriented reversely with respect to β_1 and β_2, respectively. Let a_0 be a straight arc connecting the innermost rainbow of β around z to the above small circle. It induces an arc connecting γ_1 to γ_2, still denoted by a_0. Let γ_3 be obtained by a band sum of γ_1 and γ_2 along a_1, with the induced orientation.

(5) Let H be the handlebody compatible with the diagram $(\Sigma,\{\alpha_1,\alpha_2\},\emptyset)$ and let

$$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3.$$

Rainbows and stripes are defined similarly for sutures.

The main goal is to prove the following theorem.

Theorem 3.2.19. *Suppose (H,γ) is the (1,1)-sutured-handlebody of $W(p,q,r,s)$ constructed in Construction 3.2.18. Then we have*

$$\dim_{\mathbb{C}} \underline{\mathrm{SHI}}(-H,-\gamma) \le p.$$

Before proving this theorem, we first use it to derive Theorem 1.1.5.

Proof of Theorem 1.1.5. Combining Theorem 1.1.1, Proposition 3.2.17, and Theorem 3.2.19, for a $(1,1)$-knot $K = W(p,q,r,s)$ in a lens space Y, we have

$$\dim_{\mathbb{C}} KHI(-Y,K) \le \dim_{\mathbb{C}} \underline{\mathrm{SHI}}(-H,-\gamma) \le p = \mathrm{rk}_{\mathbb{Z}} \widehat{HFK}(-Y,K) = \dim_{\mathbb{F}_2} \widehat{HFK}(-Y,K).$$

Then the theorem follows from Proposition 3.2.16, *i.e.* the mirror knot of a $(1,1)$-knot is still a $(1,1)$-knot with the same intersection number p. □

Proof of Theorem 3.2.19. We prove the theorem by induction on p for any (1,1)-diagram of $W(p,q,r,s)$ where β has only one component. This includes the case that β represents a trivial homology class. The induction is based on the bypass exact triangle in Theorem 2.3.38. We will show three balanced sutured manifolds in the bypass exact triangle are all (1,1)-sutured handlebodies, where one is the (1,1)-sutured handlebody we want and the other two are (1,1)-sutured handlebodies with smaller number p. By straightforward algebra, if the dimension inequality holds for two terms in the bypass exact triangle, then it also holds for the third term.

For the base case, consider $p = 1$. The curves $\alpha_1, \alpha_2, \beta_1, \beta_2$ in Construction 3.2.18 satisfy

$$|\alpha_1 \cap \beta_1| = |\alpha_2 \cap \beta_2| = 1.$$

It is straightforward to check (H,γ) is a product sutured manifold, so is $(-H,-\gamma)$. Then Theorem 2.3.15 implies

$$\dim_{\mathbb{C}} SHI(-H,-\gamma) = 1.$$

Now we deal with the case where $p > 1$. In Construction 3.2.18, the innermost rainbow around z, if exists, is oriented clockwise. Suppose δ_1 is either the innermost rainbow around z, or a stripe that is closest to z with z on its right-hand side. Suppose δ_2 is another rainbow or stripe that is closest to δ_1 and is to the left of δ_1. See Figure 3.4 for all possible cases. Compared to Figure 3.3, we have rotated the square counterclockwise by 90 degrees for the purpose of a better display.

We consider two different cases about the orientation of δ_2.

Case 1. Suppose δ_1 and δ_2 are oriented in parallel.

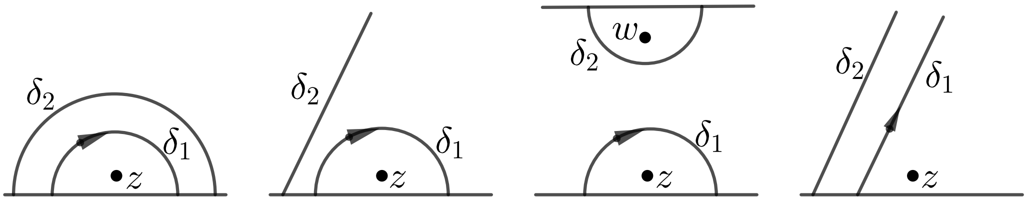


Figure 3.4 Several cases of δ_1 and δ_2.

We use $W(6,2,1,3)$ shown in Figure 3.5 as an example to carry out the proof, and the general case is similar. In this example, two innermost rainbows around z are oriented parallelly. By construction, the curve γ_3 is parallel (regardless of orientations) to γ_1 outside the neighborhood of the band-sum arc a_0. Thus, there exists a unique rainbow of γ_3 between δ_1 and δ_2 around z. Let a_1 be an anti-wave bypass arc cutting these three rainbows, as shown in Figure 3.5. Suppose γ' and γ'' are the other two sutures involved in the bypass triangle associated to a_2.

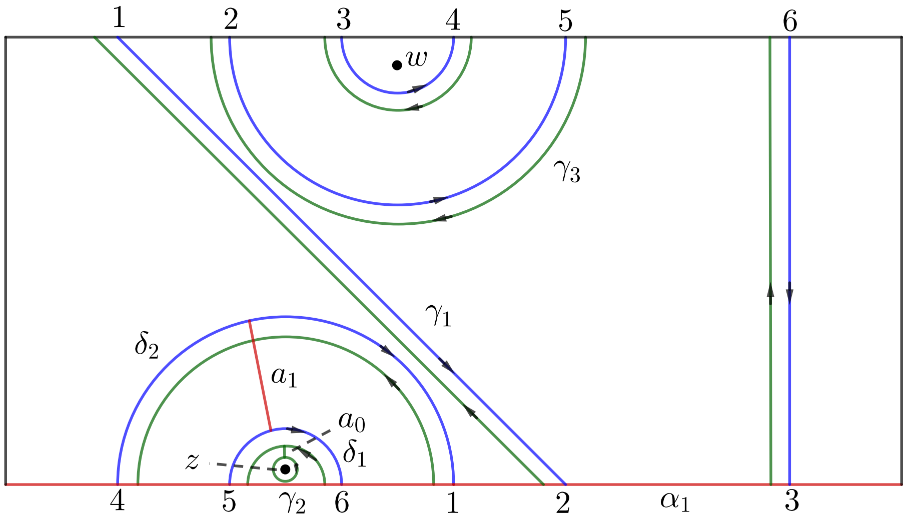


Figure 3.5 The suture related to $W(6,2,1,3)$ and the anti-wave bypass arc.

From Proposition 2.3.37, we can describe the sutures γ' and γ'' as follows. First, let γ_1^1 and γ_1^2 be two components of $\gamma_1 \backslash \partial a_1$. Suppose

$$\gamma_1' = \gamma_1^1 \cup a_1 \text{ and } \gamma_1'' = \gamma_1^2 \cup a_1$$

as shown in Figure 3.6. Second, γ' is obtained from γ by a Dehn twist along γ_1'', and γ'' is obtained from γ by a Dehn twist along γ_1'.

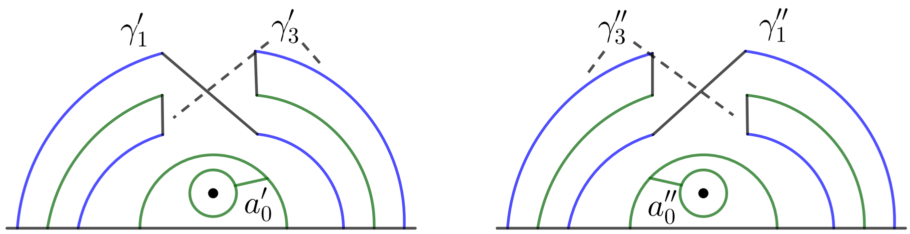


Figure 3.6 Local diagrams after bypass attachments.

There is a more direct way to describe γ' and γ''. First, note that the suture γ_2 is disjoint from both Dehn-twist curves γ_1' and γ_1'', so γ_2 remains the same in γ' and γ''. Second, it is straightforward to check the result of γ_1 under the Dehn twist along γ_1'' is γ_1', and the result of γ_1 under the Dehn twist along γ_1' is γ_1''. Thus, γ_1' is a component of γ' and γ_1'' is a component of γ''.

To figure out the image γ_3' of γ_3 under the Dehn twist along γ_1'', we first observe that we can isotop the band-sum arc a_0 to a new position a_0' such that its endpoints $\partial a_0'$ lie on $\gamma_1' \cap \gamma_1$ and γ_2, as shown in the left subfigure of Figure 3.6. Thus, the facts that a_0' is disjoint from γ_1'' and that γ_1' is the image of γ_1 under the Dehn twist along γ_1'' imply that performing a Dehn twist along γ_1'' and performing the band sum along a_0' commute with each other. Thus, we conclude that γ_3' can be obtained from a band sum on γ_1' and γ_2 along the arc a_0'. Similarly we can describe the image γ_3'' of γ_3 under the Dehn twist along γ_1'. Thus, we have described the sutures

$$\gamma' = \gamma_1' \cup \gamma_2 \cup \gamma_3' \text{ and } \gamma'' = \gamma_1' \cup \gamma_2 \cup \gamma_3''$$

explicitly, and it follows that (H,γ') and (H,γ'') are both (1,1)-sutured handlebodies. Suppose they are associated to $W(p',q',r',s')$ and $W(p'',q'',r'',s'')$, respectively.

From the above description, both γ_1' and γ_1'' are reduced. We have

$$p' + p'' = |\gamma_1' \cap \alpha_1| + |\gamma_1'' \cap \alpha_1| = |\gamma_1 \cap \alpha_1| = p.$$

Thus, the induction applies.

Case 2. Suppose δ_1 and δ_2 are oriented oppositely.

An example $W(10,3,1,5)$ is shown in Figure 3.7. By construction, there is a rainbow of γ_3 to the right of δ_2. Let a_2 be a wave bypass arc cutting δ_1, δ_2, and this rainbow as shown in Figure 3.7. Suppose γ' and γ'' are the other two sutures involved in the bypass triangle associated to a_2, respectively.

To describe the sutures γ' and γ'' more explicitly, note that the arc a_2 cuts γ_1 into two parts γ_1^1 and γ_1^2. Suppose that near a_2, γ_1^1 is to the left of a_2 and γ_1^2 is to the right of a_2. For

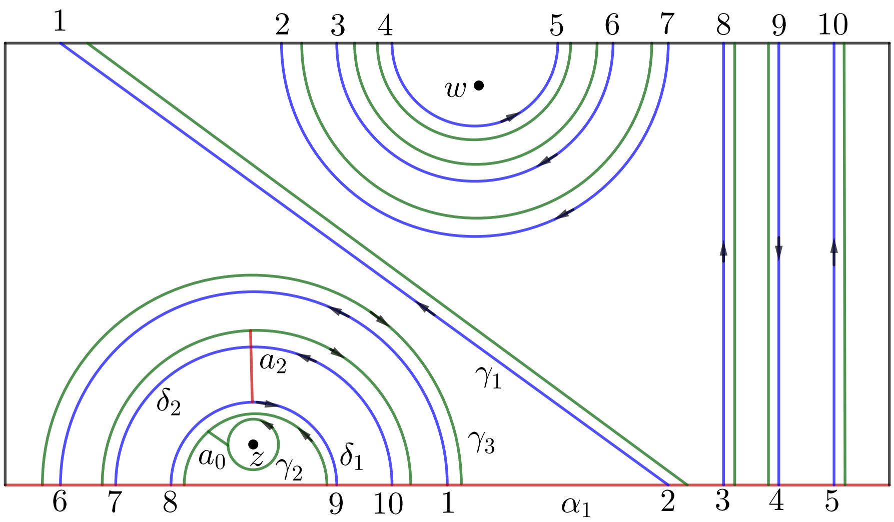


Figure 3.7 The suture related to $W(10,3,1,5)$ and the wave bypass arc.

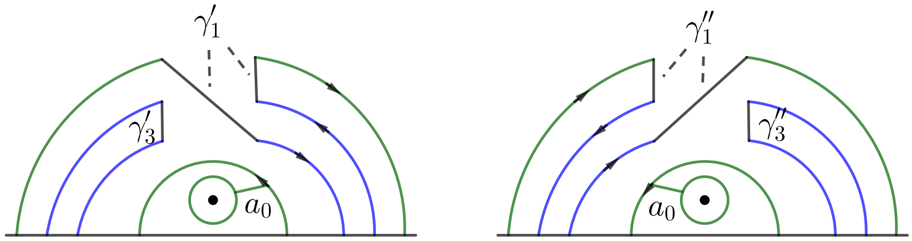


Figure 3.8 Local diagrams after bypass attachments.

γ', it consists of three components:

$$\gamma' = \gamma_1' \cup \gamma_2 \cup \gamma_3',$$

where γ_2 is as before, γ_1' is obtained by cutting γ_3 open by a_2 and gluing it to γ_1^2, and γ_3' is obtained by gluing a copy of a_2 to γ_1^1. They are depicted as in the left subfigure of Figure 3.8. Note that the curve γ_1' is not reduced. We can isotop the curve along the arc γ_1^2 into a reduced curve. The orientations of curves imply this reduced curve is depicted as in the left subfigure of Figure 3.9. Note that γ_3' is also not reduced. However, from Figure 3.9 it is straightforward to check that γ_3' can be thought of as obtained from γ_2 and γ_1' by a band sum along the arc a_0'. Also, it is clear that

$$|\gamma_1' \cap \alpha_1| = |\gamma_1^1 \cap \alpha_1|.$$

Figure 3.9 Local diagrams after isotopy.

Similarly, γ'' consists of three components:

$$\gamma'' = \gamma_1'' \cup \gamma_2 \cup \gamma_3'',$$

where γ_2 is as before, γ_1'' is obtained by cutting γ_3 open by a_2 and gluing it to γ_1^1, and γ_3'' is obtained by gluing a copy of a_2 to γ_1^2. They are depicted as in the right subfigure of Figure 3.8. Considering the orientations, we can isotop γ_2'' along γ_2^1 to the position shown in the right subfigure of Figure 3.9. Then γ_3'' can be thought of as obtained from γ_1 and γ_2'' by a band sum along a_0''. Also,

$$|\gamma_1'' \cap \alpha_1| = |\gamma_1^2 \cap \alpha_1|.$$

Hence we conclude that (H, γ') and (H, γ'') are both (1,1)-sutured-handlebodies, and

$$|\gamma_2' \cap \alpha_1| + |\gamma_2'' \cap \alpha_1| = |\gamma_{2,1} \cap \alpha_1| + |\gamma_{2,2} \cap \alpha_1| = |\gamma_2 \cap \alpha_1|.$$

Thus, the induction applies.

$\square$

Chapter 4

Calculation by Euler characteristics

In this chapter, we identify various versions of Euler characteristics of sutured instanton homology with ones of sutured Floer homology.

In the first section, we deal with the graded Euler characteristics which correspond to nontorsion part of the grading (*i.e.* the $\mathbb{Z}^{b_1(M)}$-grading for a balanced sutured manifold (M,γ)) and prove Theorem 1.2.2.

In the second section, we construct the enhanced Euler characteristic of sutured instanton homology which correspond to the full part of the $H_1(M)$-grading and prove Theorem 1.2.1. Note that an analogous construction for SFH will recover the original Euler characteristic with respect to the spin^c decomposition in (1.2.2).

In the third section, we introduce a new family of $(1,1)$-knots called **constrained knots**, whose knot Floer homology are determined by the Turaev torsion of the knot complements. Hence, combining Theorem 1.1.5, we know their instanton knot homology have the same ranks as the knot Floer homology (Corollary 1.2.9).

4.1 Graded Euler characteristics

To comparing the graded Euler characteristics of SHI and SFH, we should consider the following refinements. Suppose (M,γ) is a balanced sutured manifold and (Y,R,ω) is a closure of (M,γ) in the sense of Theorem 2.3.10. Sometimes we will omit ω and call (Y,R) a closure. Suppose $g=g(R)$ is a large and fixed number for the genus of the closure so that all given properly embedded surfaces in (M,γ) can induce surfaces in the closures as in [Li19]. Suppose $\mathbf{SHF}^g(M,\gamma)$ is the untwisted refinement of $SHI(M,\gamma)$ in [BS15, Section 9.4], which depends on the choice of g. We can also define $\chi_{\rm gr}(\mathbf{SHI}(M,\gamma))$ following Definition 2.3.30. Recall $\underline{\rm SHI}(M,\gamma)$ is the twisted refinement of $SHI(M,\gamma)$ in [BS15, Section 9.2], independent of the choice of g. From [BS15, Theorem 9.20], the restrictiono of the projectively transitive system $\underline{\mathrm{SHI}}(M,\gamma)$ on the closures with fixed genus is identified with $\mathbf{SHI}^g(M,\gamma)$, which implies

$$\chi_{\mathrm{gr}}(\underline{\mathrm{SHI}}(M,\gamma)) = \chi_{\mathrm{gr}}(\mathbf{SHI}^g(M,\gamma)). \tag{4.1.1}$$

From the discussion in Appendix A (especially, Corollary A.2.16), when considering graded Euler characteristics, we may replace $SFH(M,\gamma)$ by another equivalent version of sutured Floer homology $\mathbf{SHF}^g(M,\gamma)$ (*c.f.* Definition A.2.3 and Remark A.2.4), i.e.,

$$\chi_{\mathrm{gr}}(SFH(M,\gamma)) = \chi_{\mathrm{gr}}(\mathbf{SHF}^g(M,\gamma)). \tag{4.1.2}$$

The definition of the latter homology also depends on the genus g since it is an untwisted refinement of some sutured Floer homology $SHF(M,\gamma)$, so we temporarily use $\mathbf{SHF}^g(M,\gamma)$ to denote the dependence.

Throughout this section, we use $\mathbf{H}(Y|R)$ to denote both $I^\omega(Y|R)$ and $HF(Y|R)$ (*c.f.* Definition 2.3.4 and (A.1.4)) and use $\mathbf{SH}^g(M,\gamma)$ to denote both $\mathbf{SHI}^g(M,\gamma)$ and $\mathbf{SHF}^g(M,\gamma)$. These notations are not standard. Indeed, in [LY21b, Section 2], the notation $\mathbf{H}$ is used for "Floer-type theory", which is any $(3+1)$-TQFT satisfying some axioms and the notation $\mathbf{SH}^g$ is used for the "formal sutured homology" associated to $\mathbf{H}$. Both instanton theory and Heegaard Floer theory can be modified to satisfy the axioms, and hence both $\mathbf{SHI}^g$ and $\mathbf{SHF}^g$ can be regarded as special cases of formal sutured homology. However, in this dissertation, we do not introduce the axioms and only focus on these two specific homologies $\mathbf{SHI}^g$ and $\mathbf{SHF}^g$. Then "independent of the choice of $\mathbf{SH}^g$" means the homologies $\mathbf{SHI}^g$ and $\mathbf{SHF}^g$ give the same result. When we state a property of $\mathbf{SH}^g$, it means that both $\mathbf{SHI}^g$ and $\mathbf{SHF}^g$ have such property. Also, we use $\mathbb{F}$ to denote either $\mathbb{C}$ or $\mathbb{F}_2$, depending on the choice of $\mathbf{SH}^g$.

From Remark 2.3.31, if the admissible surfaces and the closure (Y,R,ω) of (M,γ) are fixed, then the graded Euler characteristic $\chi_{\mathrm{gr}}(\mathbf{SH}^g(M,\gamma))$ in Definition 2.3.30 and Definition A.2.14 is considered as a well-defined element

$$\chi_{\mathrm{gr}}(\mathbf{H}(Y|R)) \in \mathbb{Z}[H_1(M)/\mathrm{Tors}],$$

rather than $\mathbb{Z}[H_1(M)/\mathrm{Tors}]/\pm(H_1(M)/\mathrm{Tors})$.

4.1.1 Balanced sutured handlebodies

In this subsection, we deal with $\mathbb{Z}^n$-gradings for a balanced sutured handlebody. To be clear, we avoid using H to denote $H_1(M)/\mathrm{Tors}$ and the symbol H usually denotes a handlebody. We start with the following lemma about the sign ambiguity.

Lemma 4.1.1. *Suppose (M,γ) is a balanced sutured manifold, $S \subset (M,\gamma)$ is an admissible surface. Suppose (Y_1,R_1) and (Y_2,R_2) are two closures of (M,γ) of the same genus so that S extends to closed surfaces $\bar{S}_1$ and $\bar{S}_2$ as in Theorem 2.3.20. If $\chi_{\mathrm{gr}}(\mathbf{H}(Y_1|R_1))$ is already determined without the sign ambiguity, then $\chi_{\mathrm{gr}}(\mathbf{H}(Y_2|R_2))$ is determined without the sign ambiguity from $\chi_{\mathrm{gr}}(\mathbf{H}(Y_1|R_1))$ and the topological data of (Y_1,R_1) and (Y_2,R_2).*

Proof. From [BS15, Section 5.1], there is a canonical map

$$\Phi_{12} : \mathbf{H}(Y_1|R_1) \to \mathbf{H}(Y_2|R_2)$$

constructed by a composition of a few cobordism maps and the inverses of cobordism maps. Then the $\mathbb{Z}_2$-grading shifts follow from the degree formula (2.3.1), which only depends on the topological data of $(Y_1,R_1),(Y_2,R_2)$, and the cobordisms. By naturality, it is independent of the cobordism maps. Note that here we assume the absolute $\mathbb{Z}_2$ grading on $HF^+(Y)$ for a closed 3-manifold Y is characterized similarly to $I^\omega(Y)$. From the construction of the $\mathbb{Z}$-grading associated to S in [Li19], the canonical map Φ_{12} also preserves the grading. □

Next, we consider gradings associated to admissible surfaces. To fix the ambiguity of $H_1(M)/\mathrm{Tors}$, we will fix the choices of admissible surfaces. For sutured handlebodies, we start with embedded disks.

Proposition 4.1.2. *Suppose H is a genus $n > 0$ handlebody and $\gamma \subset \partial H$ is a closed oriented 1-submanifold so that (H,γ) is a balanced sutured manifold. Pick $D_1,\dots,D_n$ a set of pairwise disjoint meridian disks in H so that $[D_1],\dots,[D_n]$ generate $H_2(H,\partial H)$. Then for any fixed multi-grading $\boldsymbol{i} = (i_1,\dots,i_n) \in \mathbb{Z}^n$ associated to $D_1,\dots,D_n$, the Euler characteristic*

$$\chi(\mathbf{SH}^g(-H,-\gamma,\boldsymbol{i})) \in \mathbb{Z}/\{\pm 1\}$$

depends only on (H,γ), $D_1,\dots,D_n$ and $\boldsymbol{i} \in \mathbb{Z}^n$, and is independent of $\mathbf{SH}^g$. Furthermore, if a particular closure of $(-H,-\gamma)$ is fixed, then the sign ambiguity can be removed.

Proof. We fix the handlebody H and the set of disks $D_1,\dots,D_n \subset H$. For any suture γ on ∂H, define

$$I(\gamma) = \min_{\gamma' \text{ is isotopic to } \gamma} \sum_{j=1}^{g} |D_j \cap \gamma'|,$$

where $|\cdot|$ denotes the number of points. We prove the proposition by induction on $I(\gamma)$. Since $[\gamma] = 0 \in H_1(\partial H)$, we know $|D_j \cap \gamma|$ is always even for $j = 1,\dots,n$.

Note that an isotopy of γ can be understood as combinations of positive and negative stabilizations in the sense of Definition 2.3.23, and the grading shifting behavior under such isotopies (stabilizations) is described by Proposition 2.3.26, which is determined purely by topological data and is independent of $\mathbf{SH}^g$. Hence we can assume that the suture γ has already realized $I(\gamma)$.

First, if $I(\gamma) < 2n$, then there exists a meridian disk D_j with $D_j \cap \gamma = \emptyset$. Then it follows Theorem 2.3.13 that $\mathbf{SH}^g(-H, -\gamma) = 0$ since $-H$ is irreducible while $(-H, -\gamma)$ is not taut. Hence for any multi-grading $\boldsymbol{i} \in \mathbb{Z}^n$, we have $\chi(\mathbf{SH}^g(-H, -\gamma, \boldsymbol{i})) = 0$.

If $I(\gamma) = 2n$, then either there exists some integer j so that $D_j \cap \gamma = \emptyset$ or for $j = 1, \dots, n$, we have $|D_j \cap \gamma| = 2$. In the former case, we know that $\mathbf{SH}^g(-H, -\gamma) = 0$ and hence $\chi(\mathbf{SH}^g(-H, -\gamma, \boldsymbol{i})) = 0$ for any multi-grading $\boldsymbol{i} \in \mathbb{Z}^n$. In the latter case, we know that $(-H, -\gamma)$ is a product sutured manifold. It follows from Theorem 2.3.15 and Theorem 2.3.20 that

$$\mathbf{SH}^g(-H, -\gamma) = \mathbf{SH}^g(-H, -\gamma, \mathbf{0}) \cong \mathbb{F}.$$

Hence

$$\chi(\mathbf{SH}^g(-H, -\gamma, \boldsymbol{i})) = \begin{cases} \pm 1 & \boldsymbol{i} = \mathbf{0} = (0, \dots, 0) \\ 0 & \boldsymbol{i} \in \mathbb{Z}^n \backslash \{\mathbf{0}\} \end{cases}$$

Note that the ambiguity ± 1 comes from the choice of the closure. If we choose a particular closure Y of $(-H, -\gamma)$, then the Euler characteristic has no sign ambiguity. Since (H, γ) is a product sutured manifold, there is a 'standard' closure $(Y, R, \omega) = (S^1 \times \Sigma, \{1\} \times \Sigma, S^1 \times \{\mathrm{pt}\})$ as in [KM10b]. By the characterization of $\mathbb{Z}_2$-grading in Subsection 2.3.1, we have

$$\chi(\mathbf{H}(S^1 \times \Sigma | \{1\} \times \Sigma)) = -1.$$

Then for any other closure (Y, R), by Lemma 4.1.1 $\chi_{\mathrm{gr}}(\mathbf{SH}^g(Y|R))$ has no sign ambiguity.

Now suppose we have proved that, for all γ so that $I(\gamma) < 2m$, the Euler characteristic of $\mathbf{SH}^g(-H, -\gamma, \boldsymbol{i})$, viewed as an element in $\mathbb{Z}/\{\pm 1\}$, is independent of $\mathbf{SH}^g$, and that when we choose any fixed closure of $(-H, -\gamma)$, the sign ambiguity can be removed. Next we deal with the case when $I(\gamma) = 2m$.

Note that we have dealt with the base case $I(\gamma) \le 2n$, so we can assume that $m \ge n + 1$. Hence, without loss of generality, we can assume that $|D_1 \cap \gamma| \ge 4$. Within a neighborhood of ∂D_1, the suture γ can be depicted as in Figure 4.1. We can pick the bypass arc α as shown in the same figure. From Proposition 2.3.39, for any multi-grading $\boldsymbol{i} \in \mathbb{Z}^n$, we have an exact triangle

$$\begin{array}{ccc} & \mathbf{SH}^g(-H, -\gamma, \boldsymbol{i}) & \\ \nearrow & & \searrow \\ \mathbf{SH}^g(-H, -\gamma'', \boldsymbol{i}) & \longleftarrow & \mathbf{SH}^g(-H, -\gamma', \boldsymbol{i}) \end{array} \tag{4.1.3}$$

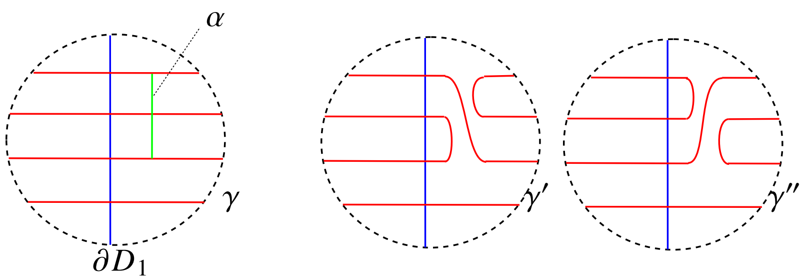


Figure 4.1 The bypass arc α that reduces the intersection function I.

Note that the suture γ' and γ'' are determined by the original suture γ and the bypass arc α, which are all topological data. From Figure 4.1, it is clear that

$$I(\gamma') \leq I(\gamma) - 2 \text{ and } I(\gamma'') \leq I(\gamma) - 2.$$

Hence the inductive hypothesis applies, and we know that the Euler characteristics of $\mathbf{SH}^g(-H, -\gamma'', \boldsymbol{i})$ and $\mathbf{SH}^g(-H, -\gamma', \boldsymbol{i})$ can be fixed independently of $\mathbf{H}$. Note that the maps in the bypass exact triangle (4.1.3) are also induced by cobordism maps (*c.f.* [BS16a, Theorem 5.2] and [BS22, Theorem 1.20]). Hence we conclude that the Euler characteristic of $\mathbf{SH}^g(-H, -\gamma, \boldsymbol{i})$ is also independent of $\mathbf{SH}^g$. Thus, we finish the proof by induction. □

Next, we deal with gradings associated to general admissible surfaces.

Proposition 4.1.3. *Suppose H is a genus n handlebody, and S is a properly embedded surface in H. Suppose $\gamma \subset \partial H$ is a suture so that (H, γ) is a balanced sutured manifold and S is an admissible surface. Then the Euler characteristic*

$$\chi(\mathbf{SH}^g(-H, -\gamma, S, j)) \in \mathbb{Z}/\{\pm 1\}$$

depends only on (H, γ), S, and $j \in \mathbb{Z}$ and is independent of $\mathbf{SH}^g$. Furthermore, if we fix a particular closure of $(-H, -\gamma)$, then the sign ambiguity can also be removed.

Before proving the proposition, we need the following lemma.

Lemma 4.1.4. *Suppose (M, γ) is a balanced sutured manifold and $S \subset (M, \gamma)$ is a properly embedded admissible surface. Suppose α is a boundary component of S so that α bounds a disk $D \subset \partial M$ and $|\alpha \cap \gamma| = 2$. Let S' be the surface obtained by taking the union $S \cup D$ and then pushing D into the interior of M. Then for any $i \in \mathbb{Z}$, we have*

$$\mathbf{SH}^g(M, \gamma, S, i) = \mathbf{SH}^g(M, \gamma, S', i).$$

Proof. Push the interior of D into the interior of M and make $D \cap S' = \emptyset$. It is clear that

$$[S] = [S' \cup D] \in H_2(M, \partial M) \text{ and } \partial S = \partial(S' \cup D).$$

In Subsection 2.3.3, when constructing the grading associated to $S' \cup D$, we can pick a closure (Y, R) of (M, γ), so that S' and D extend to closed surfaces $\bar{S}'$ and $\bar{D}$ in Y, respectively. Since $|\partial D \cap \gamma| = 2$, we know that $\bar{D}$ is a torus. Since $\partial S = \partial(S' \cup D)$, we know that S also extends to a closed surface $\bar{S}$ and from the fact that $[S] = [S' \cup D]$ we know that

$$[\bar{S}] = [\bar{S}' \cup \bar{D}] = [\bar{S}'] + [\bar{D}].$$

Since $\bar{D}$ is a torus, we know that the decompositions of $\mathbf{H}(Y|R)$ with respect to $\bar{S}$ and $\bar{S}'$ are the same. Thus it follows that

$$\mathbf{SH}^g(M, \gamma, S, i) = \mathbf{SH}^g(M, \gamma, S', i).$$

□

Proof of Proposition 4.1.3. It is a basic fact that the map

$$\partial_* : H_2(H, \partial H) \to H_1(\partial H)$$

is injective, and $H_2(H, \partial H)$ is generated by n meridian disks, which we fix as $D_1, \dots, D_n$. Hence we assume that

$$[S] = a_1[D_1] + \cdots + a_n[D_n] \in H_2(H, \partial H).$$

Case 1. ∂S consists of only ∂D_i, *i.e.*,

$$\partial S = \bigcup_{i=1}^{n} (\cup_{a_i} \partial D_i),$$

where $\cup_{a_i} \partial D_i$ means the union of a_i parallel copies of ∂D_i.

Then it follows immediately from the construction of the grading that

$$\begin{aligned} \mathbf{SH}^g(-H, -\gamma, S, j) &= \mathbf{SH}^g(-H, -\gamma, \bigcup_{i=1}^{n} (\cup_{a_i} D_i), j) \\ &= \bigoplus_{j_1 + \cdots + j_n = j} \mathbf{SH}^g(-H, -\gamma, (D_1, \dots, D_n), (j_1, \dots, j_n)). \end{aligned}$$

Hence this case follows from Proposition 4.1.2.

Case 2. ∂S contains some component that is not parallel to ∂D_i for $j = 1,\ldots,n$.

Step 1. We modify S and show that it suffices to deal with the case when $S \cap D_j = \emptyset$ for $j = 1,\ldots,n$.

Note that $\mathrm{im}(\partial_*) \subset H_1(\partial H)$ is generated by $[\partial D_1],\ldots,[\partial D_n]$, so we have $\partial S \cdot \partial D_i = 0$ for $j = 1,\ldots,n$. Here $\cdot$ denotes the algebraic intersection number of two oriented curves on ∂H. This means that for $j = 1,\ldots,n$, the intersection points of ∂D_i with ∂S can be divided into pairs. Suppose two intersection points of ∂D_1 with ∂S of opposite signs are adjacent to each other on ∂D_1, as depicted in Figure 4.2. We can perform a cut and paste surgery along D_1 and S to obtain a new surface S_1. From the same figure, it is clear that after isotopy, we can make

$$|\partial D_1 \cap \partial S_1| \leq |\partial D_1 \cap \partial S| - 2.$$

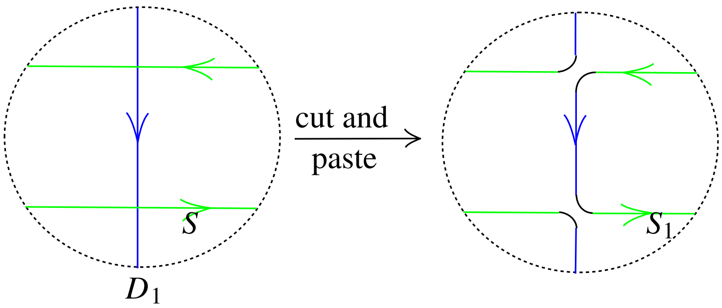


Figure 4.2 The cut and paste surgery on D_1 and S.

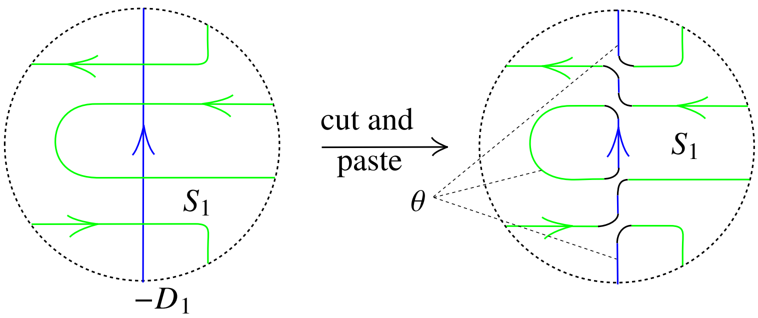


Figure 4.3 The cut and paste surgery on $-D_1$ and S_1.

Note that if we perform a cut and paste surgery along S_1 and $-D_1$, we obtain another surface S_2. From Figure 4.3 it is clear that $\partial S_2 = \partial S \cup \theta$, where θ is the union of some null-homotopic closed curves on ∂H. We can isotope S_2 to make each component of θ intersects the suture twice. Let S_3 be the resulting surface of such an isotopy and S_4 be the surface obtained from S_3 by capping off every component of θ. Then we have

$$[S] = [S_4] \in H_2(H, \partial H) \text{ and } \partial S = \partial S_4.$$

Hence from Lemma 4.1.4 we know that

$$\begin{aligned}
\mathbf{SH}^g(-H, -\gamma, S, j) &= \mathbf{SH}^g(-H, \gamma, S_4, j) \\
&= \mathbf{SH}^g(-H, \gamma, S_3, j) \\
&= \mathbf{SH}^g(-H, -\gamma, S_2, j + j(S_2, S_3)) \\
&= \bigoplus_{j_1 + j_2 = j + j(S_2, S_3)} \mathbf{SH}^g(-H, -\gamma, (D_1, S_1), (j_1, j_2))
\end{aligned}$$

By Proposition 2.3.26, the shift $j(S_2, S_3)$ depends on the isotopy from S_2 to S_3, which is determined by the topological data and is independent of $\mathbf{SH}^g$. Hence we reduce the problem to understanding the Euler characteristic of $\mathbf{SH}^g(-H, -\gamma)$ with multi-grading associated to (D_1, S_1), with

$$|\partial D_1 \cap \partial S_1| \leq |\partial D_1 \cap \partial S| - 2.$$

Repeating this argument, we finally reduce to the problem of understanding the Euler characteristic of $\mathbf{SH}^g(-H, -\gamma)$ with multi-grading associated to $(D_1, \ldots, D_n, S_n)$, with

$$\partial D_i \cap \partial S_n = \emptyset \text{ for } j = 1, \ldots, n.$$

Step 2. We modify S further to reduce to Case 1.

If every component of ∂S_n is homotopically trivial, then we know that

$$[S_n] = 0 \in H_2(H, \partial H),$$

since the map $H_2(H, \partial H) \to H_1(\partial H)$ is injective. We isotope each component of ∂S_n by stabilization to make it intersect the suture γ twice and then cap it off by a disk. The resulting surface S_{n+1} is a homologically trivial closed surface in H, so $\mathbf{SH}^g(-H, -\gamma)$ is totally supported at grading 0 with respect to S_{n+1}. The grading shift between S_n and S_{n+1} can then be understood by Proposition 2.3.26, and is independent of $\mathbf{SH}^g$.

Note that $\partial H \backslash (\partial D_1 \cup \cdots \cup \partial D_n)$ is a $2n$-punctured sphere, so ∂S is homotopically trivial when removing punctures on the sphere. If some component C of ∂S_n is not null-homotopic, then C is obtained from some ∂D_j by performing handle slides (or equivalently band sums) over $\partial D_1, \ldots, \partial D_n$ for some times.

If we isotope C to make it intersect some ∂D_i twice and then apply the cut and paste surgery, the resulting curve is isotopic to the one obtained by performing a handle slide over ∂D_i. Explicitly, in Figure 4.2, suppose two right endpoints of arcs in ∂S (the green arcs) are connected, then the right part of ∂S_1 is a trivial circle, and the left part of ∂S_1 is obtained from ∂S by performing a handle slide over ∂D_1. Thus, we can apply the cut and paste surgery for many times, which is equivalent to performing handle slides over $\partial D_1,\dots,\partial D_n$ for some times. Finally, we reduce C to the curve isotopic to ∂D_j. Then we reduce the problem to understanding the Euler characteristic of $\mathbf{SH}^g(-H,-\gamma)$ with multi-grading associated to $(D_1,\dots,D_n,S_{n+2})$, where S_{n+2} is a surface so that each component of ∂S_{n+2} is parallel to $\pm\partial D_i$ for some i. Case 1 applies to S_{n+2}, and we finish the proof. □

Corollary 4.1.5. *Suppose H is a handlebody and γ is a suture on ∂H so that (H,γ) is a balanced sutured manifold. Suppose $S_1,\dots,S_n$ are properly embedded admissible surfaces in (H,γ). Then the Euler characteristic*

$$\chi(\mathbf{SH}^g(-H,-\gamma,(S_1,\dots,S_n),(i_1,\dots,i_n)))\in\mathbb{Z}/\{\pm1\}$$

depends only on (H,γ), $S_1,\dots,S_n$, and $(i_1,\dots,i_n)\in\mathbb{Z}^n$, and is independent of $\mathbf{SH}^g$. *Furthermore, if we fix a particular closure of $(-H,-\gamma)$, then the sign ambiguity can also be removed.*

Proof. The proof is similar to that for Proposition 4.1.3. □

4.1.2 Gradings about contact 2-handle attachments

In this subsection, we prove a technical proposition about the grading behavior for the map associated to contact 2-handle attachments.

Suppose M is a compact oriented 3-manifold with boundary, and $S\subset M$ is a properly embedded surface. Suppose $\alpha\subset M$ is a properly embedded arc that intersects S transversely and $\partial\alpha\cap\partial S=\emptyset$. Let $N=M\backslash\mathrm{int}(N(\alpha))$, $S_N=S\cap N$, and $\mu\subset\partial N$ be a meridian of α that is disjoint from S_N. Let γ_N be a suture on ∂N that satisfies the following properties.

(1) (N,γ_N) is balanced, S is admissible, and $|\gamma_N\cap\mu|=2$.

(2) If we attach a contact 2-handle along μ, then we obtain a balanced sutured manifold (M,γ_M).

From Subsection 2.3.4, there is a map

$$C_\mu:\mathbf{SH}^g(-N,-\gamma_N)\to\mathbf{SH}^g(-M,-\gamma_M)$$

constructed as follows.

Push μ into the interior of N to become μ'. Suppose $(N_0,\gamma_{N,0})$ is the manifold obtained from (N,γ_N) by a 0-surgery along μ' with respect to the framing from ∂N. Equivalently, $(N_0,\gamma_{N,0})$ can be obtained from (M,γ_M) by attaching a 1-handle. Since $\mu'\subset \mathrm{int}(N)$, the construction of the closure of (N,γ_N) does not affect μ'. Thus, we can construct a cobordism between closures of (N,γ_N) and $(N_0,\gamma_{N,0})$ by attaching a 4-dimensional 2-handle associated to the surgery on μ'. This cobordism induces a cobordism map

$$C_{\mu'}:\mathbf{SH}^g(-N,-\gamma_N)\to \mathbf{SH}^g(-N_0,-\gamma_{N,0}).$$

From Subsection 2.3.4, attaching a product 1-handle does not change the closure, so there is an identification

$$\iota:\mathbf{SH}^g(-M,-\gamma_M)\xrightarrow{=}\mathbf{SH}^g(-N_0,-\gamma_{N,0}).$$

Thus, we define

$$C_\mu=\iota^{-1}\circ C_{\mu'}.$$

The main result of this subsection is the following proposition.

Proposition 4.1.6. *Consider the setting as above. For any $i\in\mathbb{Z}$, we have*

$$C_\mu(\mathbf{SH}^g(-N,-\gamma_N,S_N,i))\subset \mathbf{SH}^g(-M,-\gamma_M,S,i).$$

Proof. **Step 1**. We consider the grading behavior of the map $C_{\mu'}$ for gradings associated to S_N and S.

Since μ is disjoint from S, so we can also make μ' disjoint from $S_N=S\cap N$. As a result, the surface S_N survives in $(N_0,\gamma_{N,0})$. Thus, the cobordism map associated to the 0-surgery along μ' preserves the grading associated to S_N

$$C_{\mu'}(\mathbf{SH}^g(-N,-\gamma_N,S_N,i))\subset \mathbf{SH}^g(-N_0,-\gamma_{N,0},S_N,i).$$

Step 2. We show $\iota:\mathbf{SH}^g(-M,-\gamma_M,S,i)\xrightarrow{=}\mathbf{SH}^g(-N_0,-\gamma_{N,0},S,i)$.

As discussed above, $(N_0,\gamma_{N,0})$ is obtained from (M,γ_M) by a product 1-handle attachment. This product 1-handle can be described explicitly as follows. In $(N_0,\gamma_{N,0})$, there is an annulus A bounded by μ and its push-off μ'. We can cap off μ' by the disk coming from the 0-surgery, and hence obtain a disk D with $\partial D=\mu$. By assumption, we know that $|\partial D\cap\gamma_{N,0}|=|\mu\cap\gamma_N|=2$. Hence D is a compressing disk that intersects the suture twice. If we perform a sutured manifold decomposition on $(N_0,\gamma_{N,0})$ along D, it is straightforward to check the resulting balanced sutured manifold is (M,γ_M). However, in [Juh16], it is shown that decomposing along such a disk is the inverse operation of attaching a product 1-handle, and the disk is precisely the co-core of the product 1-handle. From this description, we can consider the product 1-handle attached to (M,γ_M) as along two endpoints of α. Since $\partial\alpha\cap\partial S=\emptyset$, the surface S naturally becomes a properly embedded surface in $(N_0,\gamma_{N,0})$. Thus, we know that the map ι preserves the gradings as claimed.

Step 3. We show $\mathbf{SH}^g(-N_0,-\gamma_{N,0},S,i)=\mathbf{SH}^g(-N_0,-\gamma_{N,0},S_N,i)$.

If $S\cap\alpha=\emptyset$, then $S=S_N=S\cap N$ and the above argument is trivial. If $S\cap\alpha\neq\emptyset$, then S_N is obtained from S by removing disks containing intersection points in $\alpha\cap S$. Then $\partial S_N\backslash\partial S$ consists of a few copies of meridians of α. For simplicity, we assume that there is only one copy of the meridian of α, *i.e.*, $\partial S_N\backslash\partial S=\mu$. The general case is similar to prove.

After performing the 0-surgery along μ', we know that the surface $S_N\subset N_0$ is compressible. Indeed, we can pick $\mu''\subset\mathrm{int}(S_N)$ parallel to $\mu\subset\partial S_N$. Then there is an annulus A' bounded by μ'' and μ', and we obtain a disk D' by capping μ' off by the disk coming from the 0-surgery. Performing a compression along the disk D', we know that S_N becomes the disjoint union of a disk D'' and the surface $S\subset N_0$. Note $\partial D''$ is parallel to the disk D discussed above. Since

$$\partial(D''\cup S)=\partial S_N \text{ and } [D''\cup S]=[S_N]\in H_2(N_0,\partial N_0),$$

From (A1-6), we know that

$$\begin{aligned}\mathbf{SH}^g(-N_0,-\gamma_{N,0},S_N,i)&=\mathbf{SH}^g(-N_0,-\gamma_{N,0},S\cup D'',i)\\&=\bigoplus_{i_1+i_2=i}\mathbf{SH}^g(-N_0,-\gamma_{N,0},(S,D''),(i_1,i_2)).\end{aligned}$$

Since the disk D'' intersects γ'_N twice, from term (2) of Theorem 2.3.20, we know that

$$\mathbf{SH}^g(-N_0,-\gamma_{N,0})=\mathbf{SH}^g(-N_0,-\gamma_{N,0},D'',0).$$

Hence we conclude that

$$\begin{aligned}\mathbf{SH}^g(-N_0,-\gamma_{N,0},S_N,i)&=\sum_{i_1+i_2=i}\mathbf{SH}^g(-N_0,-\gamma_{N,0},(S,D''),(i_1,i_2))\\&=\mathbf{SH}^g(-N_0,-\gamma_{N,0},S,i).\end{aligned}$$

□

4.1.3 General balanced sutured manifolds

In this subsection, we prove the main theorem of this section. Note that Theorem 1.2.2 follows directly from it and facts (4.1.1), (4.1.2).

Theorem 4.1.7. *Suppose* (M,γ) *is a balanced sutured manifold and* $\{S_1,\dots,S_n\}$ *is a collection of properly embedded admissible surfaces. Then the Euler characteristic*

$$\chi(\mathbf{SH}^g(-M,-\gamma,(S_1,\dots,S_n),(i_1,\dots,i_n)))$$

depends only on (M,γ)*,* $S_1,\dots,S_n$*, and* $(i_1,\dots,i_n)\in\mathbb{Z}^n$*, and is independent of* $\mathbf{SH}^g$*.*

Proof of Theorem 4.1.7. First we can attach product 1-handles disjoint from $S_1,\dots,S_n$. From Subsection 2.3.4, attaching a product 1-handle does not change the closure (note that g is large enough) and hence does not make any difference to the multi-grading associated to $(S_1,\dots,S_n)$. Hence we can assume that γ is connected from now on. From Construction 3.2.7, we can pick a disjoint union of properly embedded arcs

$$T=\theta_1\cup\cdots\cup\theta_m$$

so that

(1) for $k=1,\dots,m$, we have $\partial\theta_k\cap R_+(\gamma)\neq\emptyset$ and $\partial\theta_k\cap R_-(\gamma)\neq\emptyset$,

(2) $M_T:=M\backslash\mathrm{int}(N(T))$ is a handlebody.

We use H to denote M_T and write $S_{j,H}=S_j\cap H$. We prove the theorem in the case when $m=1$, while the general case follows from a straightforward induction. If $m=1$, then T is connected. Suppose μ is the meridian of T and suppose μ' is a push-off of μ inside H. By Lemma 3.1.8, the surgery exact triangle along μ' induces the following exact triangle

$$\begin{array}{ccccc} & & \mathbf{SH}^g(-M,-\gamma) & & \\ & \overset{C_\mu}{\nearrow} & & \searrow & \\ \mathbf{SH}^g(-H,-\Gamma_0) & & \longleftarrow & & \mathbf{SH}^g(-H,-\Gamma_1) \end{array} \qquad (4.1.4)$$

where Γ_0 and Γ_1 are constructed in Subsection 5.1.1 and the map C_μ is the map associated to the contact 2-handle attachment along μ. Since μ is disjoint from $S_{j,H}$ for $j=1,\dots,n$, the proof of Proposition 4.1.6 implies there is a graded version of the exact triangle (4.1.4):

$$\begin{array}{ccc} & & \mathbf{SH}^g(-M,-\gamma,(S_1,\dots,S_n),(i_1,\dots,i_n)) \\ & \overset{C_\mu}{\nearrow} & \downarrow \\ \mathbf{SH}^g(-H,-\Gamma_0,(S_{1,H},\dots,S_{n,H}),((i_1,\dots,i_n))) & \nwarrow & \\ & & \mathbf{SH}^g(-H,-\Gamma_1,(S_{1,H},\dots,S_{n,H}),((i_1,\dots,i_n))) \end{array} \tag{4.1.5}$$

Then Theorem 4.1.7 follows from Proposition 2.3.7 and Corollary 4.1.5. □

4.2 Enhanced Euler characteristics

In this section, we refine results in Section 3.1 to obtain a decomposition of sutured instanton homology. In Section 4.1, we used the untwisted refinement $\mathbf{SHI}^g$ to carry out proofs. However, we have to use the twisted refinement $\underline{\mathrm{SHI}}$ in this section by the following reasons:

(1) some properties involve closures of different genera (*e.g.* the surface decomposition theorem in Term (2) of Theorem 2.3.20), while the genus is fixed in $\mathbf{SHI}^g$;

(2) the proof of the functoriality of contact gluing maps in [Li18] involves closures obtained from disconnected auxillary surfaces, which can only be handled by a genus one version of Floer's excision theorem that is available in the twisted theory. Note that in Subsection 4.1.2 we only use the construction of contact gluing maps and do not use the functoriality.

Also, we do not use the untwisted refinement $\mathbf{SHF}^g$ for Heegaard Floer theory and use the original SFH instead. The discussion in Section A.2 implies SFH shares many properties with $\underline{\mathrm{SHI}}$. Hence we can either use $\underline{\mathrm{SHI}}$ or SFH in the construction of this section (*c.f.* the proof of Theorem 4.2.21). We write $\underline{\mathrm{SHG}}$ for both $\underline{\mathrm{SHI}}$ or SFH. This is also not a standard notation, since it usually denotes $\underline{\mathrm{SHI}}$ and the twisted refinement $\underline{\mathrm{SHM}}$ of sutured monopole homology. Similarly, we use $\mathbb{F}$ to denote either $\mathbb{C}$ or $\mathbb{F}_2$.

4.2.1 One tangle component

In this subsection, we apply lemmas in Section 3.1 to obtain a decomposition of $\underline{\mathrm{SHG}}$ associated to one tangle component. We adapt the notations in Subsection 3.1.1. Suppose (M,γ) is a balanced sutured manifold and suppose $T\subset(M,\gamma)$ is a vertical tangle with only one component $\alpha=T_1$, which is rationally null-homologous of order q. Let M_T be the manifold obtained from M by removing a neighborhood of T and let $\gamma_T = \gamma \cup m_\alpha$, where m_α is a positively oriented meridian of α. Suppose $S_j \subset (M_T, \Gamma_j)$ are constructed from a Seifert surface of the tangle for $j \in \mathbb{N} \cup \{+,-\}$. Recall that we use either $\mathbb{Z}$-grading or $(\mathbb{Z}+\frac{1}{2})$-grading associated to surfaces. For simplicity, we will still say a grading i is in $\mathbb{Z}$. Also, for simplicity, we write $\chi(\bar{S}_j) = \chi(S_j) - \frac{1}{2}|S_j \cap \Gamma_j|$ as in Theorem 2.3.20.

We start with the following lemma, which roughly says the summands in the 'middle' gradings of $\underline{\mathrm{SHG}}(-M_T, -\Gamma_n)$ associated to S_n are periodic of order q.

Lemma 4.2.1. *Suppose $n \in \mathbb{N}$ and $i_1, i_2 \in \mathbb{Z}$ satisfying $i_1, i_2 \in (\rho_n, P_n)$ and $i_1 - i_2 = q$, where ρ_n and P_n are constants in Lemma 3.1.7:*

$$\rho_n = i^n_{max} - nq \text{ and } P_n = i^n_{min} + (n+1)q.$$

Then we have

$$\underline{\mathrm{SHG}}(-M_T, -\Gamma_n, S_n, i_1) \cong \underline{\mathrm{SHG}}(-M_T, -\Gamma_n, S_n, i_2).$$

Proof. This follows directly from isomorphisms in Lemma 3.1.7 and the computation on gradings. □

Note that

$$\begin{aligned} P_n - \rho_n = & \ (i^n_{min} + (n+1)q) - (i^n_{max} - nq) \\ = & \ -(i^n_{max} - i^n_{min}) + (2n+1)q \\ = & \ -(-\chi(\bar{S}_+) + nq) + (2n+1)q \\ = & \ \chi(\bar{S}_+) + (n+1)q. \end{aligned}$$

Thus, the difference of P_n and ρ_n can be infinitely large.

Definition 4.2.2. Define $Q_n = P_n - q$. Suppose $n \in \mathbb{N}$ satisfies $Q_n - \rho_n > q$, define

$$\mathcal{SHG}_\alpha(-M, -\gamma, i) := \underline{\mathrm{SHG}}(-M_T, -\Gamma_n, S_n, Q_n - i),$$

and

$$\mathcal{SHG}_\alpha(-M, -\gamma) := \bigoplus_{i=1}^{q} \mathcal{SHG}_\alpha(-M, -\gamma, i).$$

Remark 4.2.3. The definition of Q_n comes from the following fact

$$i^n_{max} - Q_n = i^n_{max} - (P_n - q) = -\chi(\bar{S}_+) = \rho_n - i^n_{min} \tag{4.2.1}$$

Remark 4.2.4. From Lemma 3.1.7 and the fact

$$P_{n+1} - P_n = i^{n+1}_{min} - i^n_{min} + q = i^{n+1}_{max} - i^n_{max},$$

the isomorphism class of $\mathcal{SHG}_\alpha(-M,-\gamma,i)$ is independent of the choice of the large integer n. Also, by Lemma 4.2.1, the isomorphism class of $\mathcal{SHG}_\alpha(-M,-\gamma)$ would be the same (up to a $\mathbb{Z}_q$ grading shift) if we consider arbitrary q many consecutive gradings within the range (ρ_n, P_n).

Remark 4.2.5. For a rationally null-homologous knot $\widehat{K} \subset \widehat{Y}$ with a basepoint p, we can remove a neighborhood of p and add a suture δ on $\partial N(p)$ such that two intersection points of $\widehat{K}$ and $\partial N(p)$ lie on $R_+(\gamma)$ and $R_-(\gamma)$, respectively. Then $\widehat{K}$ becomes a vertical tangle α in $(\widehat{Y} - \text{int}N(p), \delta)$ which is rationally null-homologous. In this case, $\mathcal{SHG}_\alpha(\widehat{Y} - \text{int}N(p), \delta, i)$ reduces to $\mathcal{I}_+(-\widehat{Y}, \widehat{K}, i)$ in [LY22, Definition 4.21], up to a $\mathbb{Z}_q$ grading shift.

In the rest of this subsection, we will show that there is an isomorphism

$$\mathcal{SHG}_\alpha(-M,-\gamma) \cong \underline{\mathrm{SHG}}(-M,-\gamma).$$

Hence the decomposition of $\mathcal{SHG}_\alpha(-M,-\gamma)$ provides a decomposition of $\underline{\mathrm{SHG}}(-M,-\gamma)$. To do so, we first show their dimensions are the same and then show there is a surjective map from $\mathcal{SHG}_\alpha(-M,-\gamma)$ to $\underline{\mathrm{SHG}}(-M,-\gamma)$.

For simplicity, we introduce the following notions.

Definition 4.2.6. Suppose $n \in \mathbb{N}$. The direct sum of some consecutive gradings of

$$\underline{\mathrm{SHG}}(-M_T, -\Gamma_n, S_n)$$

is called a **block**. For a block A, the number of gradings involved is called the **size** of A.

Example 4.2.7. Suppose $n \in \mathbb{N}$ satisfies $Q_n - \rho_n > q$. Let A, B, C and D be the blocks consisting of the top $(-\chi(\bar{S}_+)+1)$ gradings, the next q gradings, the next

$$(i^n_{max} - i^n_{min} + 1) - 2(-\chi(\bar{S}_+)+1) - q = \chi(\bar{S}_+) + (n-1)q - 1$$

gradings, and the last $(-\chi(\bar{S}_+)+1)$ gradings of $\underline{\mathrm{SHG}}(-M_T, -\Gamma_n, S_n)$, respectively. We write

$$\underline{\mathrm{SHG}}(-M_T, -\Gamma_n, S_n) = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}.$$

From Definition 4.2.2 and fact (4.2.1), we know that $\mathcal{SHG}_{\alpha}(-M,-\gamma)$ is the block B. Also, we can write

$$\underline{\mathrm{SHG}}(-M_T,-\Gamma_n,S_n)=\begin{pmatrix}A\\E\\F\\D\end{pmatrix},$$

where E and F are of size $(\chi(\bar{S}_+)+(n-1)q-1)$ and q, respectively. By comparing the gradings, we have

$$\begin{pmatrix}B\\C\end{pmatrix}=\begin{pmatrix}E\\F\end{pmatrix}$$

Note that we do not have $B=E$ and $C=F$ since they have different sizes. However, when putting together, the total size of B and C equals that of E and F.

Lemma 4.2.8. *Let $\mathcal{SHG}_{\alpha}(-M,-\gamma)$ be defined as in Definition 4.2.2. We have*

$$\dim_{\mathbb{F}}\mathcal{SHG}_{\alpha}(-M,-\gamma)=\dim_{\mathbb{F}}\underline{\mathrm{SHG}}(-M,-\gamma).$$

Proof. Suppose $n\in\mathbb{N}$ satisfies $Q_n-\rho_n>q$. We can apply Proposition 3.1.6. Using blocks, we have the following. There is not enough room for writing down the whole notations, so we will only write down the sutures to denote them.

size	$\Gamma_+ \xrightarrow{\psi^+_{+,n}}$	$\Gamma_n \xrightarrow{\psi^n_{+,n+1}}$	$\Gamma_{n+1} \xrightarrow{\psi^{n+1}_{+,+}}$	Γ_+
q	G		X_1	G
$-\chi(\bar{S}_+)+1$	H	A	X_2	H
$\chi(\bar{S}_+)+(n-1)q-1$		E	X_3	
q		F	X_4	
$-\chi(\bar{S}_+)+1$		D	X_5	

The empty block implies the summands in the block are zeros. Note that

$$i^+_{max}-i^+_{min}+1=-\chi(\bar{S}_+)+1\leq q+(-\chi(\bar{S}_+)+1).$$

From the exactness, we know that

$$X_1=G,\ X_3=E,\ X_4=F,\ \text{and}\ X_5=D.$$

There is another bypass exact triangle, and similarly we have

$$\begin{array}{ccccccc}
\text{size} & \Gamma_- & \xrightarrow{\psi^-_{-,n}} & \Gamma_n & \xrightarrow{\psi^n_{-,n+1}} & \Gamma_{n+1} & \xrightarrow{\psi^{n+1}_{-,-}} & \Gamma_- \\
-\chi(\bar{S}_+)+1 & & & A & & A & & \\
q & & & B & & B & & \\
\chi(\bar{S}_+)+(n-1)q-1 & & & C & & C & & \\
-\chi(\bar{S}_+)+1 & I & & D & & X_6 & & I \\
q & J & & & & J & & J
\end{array}$$

Note that

$$i^-_{max} - i^-_{min} + 1 = -\chi(\bar{S}_+) - q + 1 \leq q + (-\chi(\bar{S}_+) + 1).$$

Comparing the two expressions of $\underline{\mathrm{SHG}}(-M_T, -\Gamma_{n+1}, S_n)$, we have

$$\begin{pmatrix} G \\ X_2 \\ E \\ F \\ D \end{pmatrix} = \underline{\mathrm{SHG}}(-M_T, -\Gamma_{n+1}, S_n) = \begin{pmatrix} A \\ B \\ C \\ X_6 \\ J \end{pmatrix}.$$

Taking sizes into consideration, we know that

$$\begin{pmatrix} G \\ X_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix},\ E = C,\ \text{and}\ \begin{pmatrix} F \\ D \end{pmatrix} = \begin{pmatrix} X_6 \\ J \end{pmatrix}.$$

Thus, we know that

$$\underline{\mathrm{SHG}}(-M_T, -\Gamma_{n+1}, S_n) = \begin{pmatrix} A \\ B \\ E \\ F \\ D \end{pmatrix}.$$

Comparing this expression with the expression of $\underline{\mathrm{SHG}}(-M_T, -\Gamma_n, S_n)$ in Example 4.2.7, we have

$$\begin{aligned}
\dim_{\mathbb{F}} \mathcal{SHG}_\alpha(-M, -\gamma) &= \dim_{\mathbb{F}} B \\
&= \dim_{\mathbb{F}} \underline{\mathrm{SHG}}(-M_T, -\Gamma_{n+1}) - \dim_{\mathbb{F}} \underline{\mathrm{SHG}}(-M_T, -\Gamma_n) \\
&= \dim_{\mathbb{F}} \underline{\mathrm{SHG}}(-M, -\gamma).
\end{aligned}$$

Note that the last equality follows from Lemma 3.1.9. □

Remark 4.2.9. The essential difference for the case of tangles is that Γ_+ is not equal to Γ_-, though it is true in the case of knots in Remark 4.2.5.

Proposition 4.2.10. *Suppose $n \in \mathbb{N}$ is large enough. Then the map F_n in Lemma 3.1.8 restricted to $\mathcal{SHG}_\alpha(-M,-\gamma)$ is an isomorphism, i.e.*

$$F_n|_{\mathcal{SHG}_\alpha(-M,-\gamma)} : \mathcal{SHG}_\alpha(-M,-\gamma) \xrightarrow{\cong} \underline{\mathrm{SHG}}(-M,-\gamma).$$

Proof. By Lemma 4.2.8, it suffices to show that the restriction of F_n is surjective. By Lemma 3.1.9, we know that F_n is surjective. Then it suffices to show that F_n remains surjective when restricted to $\mathcal{SHG}_\alpha(-M,-\gamma)$. For any $x \in \underline{\mathrm{SHG}}(-M,-\gamma)$, let $y \in \underline{\mathrm{SHG}}(-M_T,-\Gamma_n)$ be an element so that $F_n(y) = x$. Suppose

$$y = \sum_{j\in\mathbb{Z}} y_j, \text{ where } y_j \in \underline{\mathrm{SHG}}(-M_T,-\Gamma_n,S_n,j).$$

For any y_j, we want to find $y'_j \in \mathcal{SHG}_\alpha(-M,-\gamma)$ so that $F_n(y_j) = F_n(y'_j)$.

To do this, we first assume that $j \geq Q_n$. Then there exists an integer m so that

$$Q_n - q \leq j - mq \leq Q_n - 1.$$

We can take

$$y'_j = (\psi^{n,j-mq}_{-,n+1})^{-1} \circ \cdots \circ (\psi^{n,i^{n+m}_{max}-i^n_{max}+j-mq}_{-,n+m})^{-1} \circ \psi^{n+m-1}_{+,n+m} \circ \cdots \circ \psi^n_{+,n+1}(y_j). \tag{4.2.2}$$

From Lemma 3.1.7, all the negative bypass maps involved in (4.2.2) are isomorphisms so the inverses exist. Also, we have

$$y'_j \in \underline{\mathrm{SHG}}(-M_T,-\Gamma_n,S_n,j-mq) \subset \mathcal{SHG}_\alpha(-M,-\gamma).$$

Finally, from commutative diagrams in Lemma 3.1.8, we know that $F_n(y'_j) = F_n(y_j)$.

For

$$j \in [Q_n - q, Q_n - 1],$$

we can simply take $y'_j = y_j$.

For $j < Q_n - q$, we can pick y'_j similarly, while switching the roles of $\psi^*_{+,*}$ and $\psi^*_{-,*}$ in (4.2.2).

In summary, we can take

$$y' = \sum_{j\in\mathbb{Z}} y'_j \in \mathcal{SHG}_\alpha(-M,-\gamma) \text{ with } F_n(y') = F_n(y) = x.$$

Hence F_n is surjective. □

Remark 4.2.11. In Definition 4.2.2, we use a large enough integer n to define $\mathcal{SHG}_\alpha(-M,-\gamma)$. We can also define Γ_{-n} from Γ_0 by twisting along γ_1 for n times. For a large enough integer n, we can define a space $\mathcal{SHG}'_\alpha(-M,-\gamma)$ generalizing $\mathcal{I}_-(-\widehat{Y},\widehat{K})$ in [LY22, Definition 4.27]. However, from the discussion in [LY22, Section 4.4 in ArXiv version 2] between $\mathcal{I}_+(-\widehat{Y},\widehat{K})$ and $\mathcal{I}_-(-\widehat{Y},\widehat{K})$, we expect that $\mathcal{SHG}'_\alpha(-M,-\gamma)$ is isomorphic to $\mathcal{SHG}_\alpha(-M,-\gamma)$ up to a $\mathbb{Z}_q$ grading shift. Hence there is no new information and we skip the discussion here.

4.2.2 More tangle components

In this subsection, we obtain a decomposition of $\underline{\mathrm{SHG}}$ associated to more tangle components. Suppose (M,γ) is a balanced sutured manifold. For a vertical tangle T in M, let $M_T = M\backslash \mathrm{int}N(T)$ and let γ_T be the union of γ and positively oriented meridians of components of T.

First, we prove some lemmas about homology groups.

Lemma 4.2.12. *For any connected tangle α in M, we have*

$$\mathrm{rk}_{\mathbb{Z}} H_1(M_\alpha) = \begin{cases} \mathrm{rk}_{\mathbb{Z}} H_1(M) & \text{if } [\alpha] \neq 0 \in H_1(M,\partial M;\mathbb{Q}), \\ \mathrm{rk}_{\mathbb{Z}} H_1(M)+1 & \text{if } [\alpha] = 0 \in H_1(M,\partial M;\mathbb{Q}). \end{cases}$$

Proof. Consider the long exact sequence assoicated to the pair (M,M_α):

$$H^1(M,M_\alpha) \xrightarrow{p_1^*} H^1(M) \xrightarrow{i_1^*} H^1(M_\alpha) \xrightarrow{\delta_1^*} H^2(M,M_\alpha) \xrightarrow{p_2^*} H^2(M) \xrightarrow{i_2^*} H^2(M_\alpha) \xrightarrow{\delta_2^*} H^3(M,M_\alpha). \tag{4.2.3}$$

By the excision theorem, we have

$$H^*(M,M_\alpha) \cong H^j(N(\alpha),\partial N(\alpha)\cap M_\alpha) \cong H^j(D^2,\partial D^2) \cong \begin{cases} \mathbb{Z} & j=2, \\ 0 & j=1,3. \end{cases}$$

Since $H^2(N(\alpha),\partial N(\alpha)\cap M_\alpha)$ is generated by the disk that is the Poincaré dual of $[\alpha\cap N(\alpha)]$ and p_2^* is induced by the projection, the image of p_2^* is generated by the Poincaré dual of

$[\alpha]$. Since $H^1(M)$ and $H_1(M)$ always have the same rank, we obtain the rank equation from (4.2.3). □

Lemma 4.2.13. *Suppose (M,γ) is a balanced sutured manifold. There exists a (possibly empty) tangle $T = T_1 \cup \cdots \cup T_m$ in M, such that* $\mathrm{Tors}H_1(M_T) = 0$ *and for any $T' \subset T$ and $T_i \subset T \backslash T'$, we have*

$$[T_i] = 0 \in H_1(M_{T'}, \partial M_{T'}; \mathbb{Q}). \tag{4.2.4}$$

Proof. Suppose α is a connected tangle in M. From (4.2.3) and the proof of Lemma 4.2.12, we have

$$\mathbb{Z}\langle\phi_\alpha\rangle \xrightarrow{p_2^*} H^2(M) \xrightarrow{i_2^*} H^2(M_\alpha) \to 0,$$

where ϕ_α is the Poincaré dual of $[\alpha]$. By the universal coefficient theorem, the torsion subgroups of $H^2(M)$ and $H_1(M)$ are isomorphic. In particular, $\mathrm{Tors}H^2(M) = 0$ if and only if $\mathrm{Tors}H_1(M) = 0$. Let α be a rationally null-homologous tangle, then

$$\mathrm{Tors}H^2(M_\alpha) \cong \mathrm{Tors}H^2(M)/\mathrm{PD}(\alpha).$$

Thus, we can always choose connected tangles

$$T_1 \subset M, T_2 \subset M_{T_1}, T_3 \subset M_{T_1\cup T_2}, \ldots, T_m \subset M_{T_1\cup\cdots\cup T_{m-1}}$$

that are rationally null-homologous to kill the whole torsion subgroup. In other words, for $T = T_1 \cup \cdots \cup T_m$, we have $\mathrm{Tors}H_1(M_T) = 0$.

By Lemma 4.2.12, we have

$$\mathrm{rk}_{\mathbb{Z}} H_1(M_T) = \mathrm{rk}_{\mathbb{Z}} H_1(M) + m. \tag{4.2.5}$$

Hence for any T' and any T_i satisfying the assumption, (4.2.4) holds, otherwise it contradicts with the rank equality (4.2.5). □

Remark 4.2.14. Since moving the endpoints of a tangle on the boundary of the ambient 3-manifold does not change the homology class of the tangle, we can suppose the tangle T in Lemma 4.2.13 is a vertical tangle. Moreover, when M has connected boundary, we can suppose endpoints of T all lie in a neighborhood of a point on the suture γ.

Lemma 4.2.15. *Suppose (M,γ) is a balanced sutured manifold and suppose α is a connected rationally null-homologous tangle of order q. Let S_α be a Seifert surface of T_i, i.e., ∂S_i consists of q parallel copies of α and arcs on ∂M. Suppose $S_1, \ldots, S_n$ are admissible surfaces in (M,γ) generating $H_2(M,\partial M)$. Then the restrictions of $S_1, \ldots, S_n$ and S_α on M_T generate $H_2(M_T, \partial M_T)$.*

Proof. From (4.2.3) and the proof of Lemma 4.2.12, we have

$$0 \to H^1(M) \xrightarrow{i_1^*} H^1(M_\alpha) \xrightarrow{\delta_1^*} \mathbb{Z}\langle\phi_\alpha\rangle \xrightarrow{p_2^*} H^2(M),$$

where ϕ_α is the Poincaré dual of $[\alpha]$. It is straighforward to calculate

$$\delta_1^*(\mathrm{PD}([S_\alpha])) = q\phi_\alpha. \tag{4.2.6}$$

Since $H^1(M) \cong H_2(M,\partial M)$, we have

$$H_2(M_\alpha,\partial M_\alpha)/H_2(M,\partial M) \cong H^1(M_\alpha)/H^1(M) \cong H^1(M_\alpha)/\mathrm{im} i_1^* \cong H^1(M_\alpha)/\ker\delta_1^* \cong \mathrm{im}\delta_1^* \cong \ker p_2^*. \tag{4.2.7}$$

Since the image of p_2^* is the Poincaré dual of $[\alpha]$, we have

$$\ker p_2^* \cong \langle q\phi_\alpha\rangle. \tag{4.2.8}$$

Combining (4.2.6), (4.2.7), and (4.2.8), we know that $[S_\alpha]$ generates $H_2(M_\alpha,\partial M_\alpha)/H_2(M,\partial M)$. Thus, we conclude the desired property.

□

In the rest of this subsection, we suppose (M,γ) is a balanced sutured manifold and $T = T_1 \cup \cdots \cup T_m$ is a vertical tangle satisfying Lemma 4.2.13. Suppose the order of the first component T_1 in $H_1(M)$ is q_1 and suppose S_1 is a Seifert surface of T_1.

Convention. We will still use S_1 to denote its restriction on M_{T_1}. This also applies to other Seifert surfaces mentioned below.

We adapt the construction in Subsection 3.1.1. Applying results in Subsection 4.2.1, we have

$$\mathcal{SHG}_{T_1}(-M,-\gamma) := \bigoplus_{i=1}^{q_1} \underline{\mathrm{SHG}}(-M_{T_1},-\Gamma_n,(S_1)_n,Q_n - i) \cong \underline{\mathrm{SHG}}(-M,-\gamma),$$

where n is a large integer, $(S_1)_n$ is the restriction of S_1, and Q_n is a fixed integer. For simplicity, we choose a large integer n_1 such that $(S_1)_{n_1} = S_1$ and write

$$\Gamma_{n_1}^1 = \Gamma_n|_{n=n_1} \text{ and } Q_{n_1}^1 = Q_n|_{n=n_1}.$$

For the second component T_2, suppose S_2 is its Seifert surface in M_{T_1} with ∂S^2 containing q_2 copies of T_2. Now we can apply the construction in Subsection 3.1.1 and the results in

Subsection 4.2.1 to $(M,\Gamma^1_{n_1})$. For a large integer n_2 such that $(S_2)_{n_1}=S_2$, we define

$$\begin{aligned}\mathcal{SHG}_{T_1\cup T_2}(-M,-\gamma):=&\bigoplus_{i_1=1}^{q_1}\bigoplus_{i_2=1}^{q_2}\underline{\mathrm{SHG}}(-M_{T_1\cup T_2},-\Gamma^2_{n_2},(S_1,S_2),(Q^1_{n_1}-i_1,Q^2_{n_2}-i_2))\\ \cong&\underline{\mathrm{SHG}}(-M,-\gamma).\end{aligned}$$

Iterating this procedure, we have the following definition.

Definition 4.2.16. For $i=1,\dots,m$, suppose the component T_k is rationally null-homologous of order q_k in $M_{T_1\cup\cdots\cup T_{k-1}}$. Inductively, for $k=1,\dots,m$, we choose a large integer n_k, a suture $\Gamma^k_{n_k}\subset\partial M_{T_1\cup\cdots\cup T_k}$, a Seifert surface $S_k=(S_k)_{n_k}\subset M_{T_1\cup\cdots\cup T_k}$, and an integers $Q^k_{n_k}$, such that $n_k,\Gamma^k_{n_k},S_k,Q^k_{n_k}$ depend on the choices for the first $(k-1)$ tangles. Define

$$\mathcal{SHG}_T(-M,-\gamma):=\bigoplus_{i_1\in[1,q_1],\dots,i_m\in[1,q_m]}\underline{\mathrm{SHG}}(-M_T,-\Gamma^m_{n_m},(S_1,\dots,S_m),(Q^1_{n_1}-i_1,\cdots,Q^m_{n_m}-i_m)).$$

Remark 4.2.17. Though we only use the subscript T in the notation $\mathcal{SHG}_T(-M,-\gamma)$, it is not known if $\mathcal{SHG}_T(-M,-\gamma)$ is independent of the choices of all constructions. In particular, we have to choose an order of the components to define $\mathcal{SHG}_T(-M,-\gamma)$.

Applying results in Subsection 4.2.1 for m times, the following proposition is straightforward.

Proposition 4.2.18. $\mathcal{SHG}_T(-M,-\gamma)\cong\underline{\mathrm{SHG}}(-M,-\gamma)$.

The map $H_1(M_{T_1})\to H_1(M)$ is surjective. The q_1 direct summands of $\underline{\mathrm{SHG}}_{T_1}(-M,-\gamma)$ correspond to the order q_1 torsion subgroup generated by

$$[T_1]\in\mathrm{Tors}H_1(M,\partial M)\cong\mathrm{Tors}H^2(M)\cong\mathrm{Tors}H_2(M)$$

Hence the summands of $\underline{\mathrm{SHG}}_{T_1}(-M,-\gamma)$ provide a decomposition of $\underline{\mathrm{SHG}}(-M,-\gamma)$ with respect to the torsion subgroup generated by $[T_1]$. By induction and the fact that $\mathrm{Tors}H_1(M_T)=0$, we can regard summands in $\mathcal{SHG}_T(-M,-\gamma)$ as a decomposition of $\underline{\mathrm{SHG}}(-M,-\gamma)$ with respect to $\mathrm{Tors}H_1(M)$.

To provide a decomposition of $\underline{\mathrm{SHG}}(-M,-\gamma)$ with respect to the whole $H_1(M)$ as in Theorem 1.2.1, we choose admissible surfaces $S_{m+1},\dots,S_{m+n}$ generating $H_2(M,\partial M)$. By Lemma 4.2.15, the restrictions of $S_1,\dots,S_{m+n}$ generate $H_2(M_T,\partial M_T)$. By Proposition 4.1.6 (the result also applies to the twisted refinement), the gradings associated to these surfaces behave well under restriction.

Definition 4.2.19. Consider the construction as above. For $i = 1, \dots, m+n$, let $\rho_1, \dots, \rho_{m+n} \in H_1(M_T) = H_1(M_T)/\mathrm{Tors}$ be the class satisfying $\rho_i \cdot S_j = \delta_{i,j}$. Consider

$$j_* : \mathbb{Z}[H_1(M_T)] \to \mathbb{Z}[H_1(M)].$$

We write

$$H = H_1(M), \boldsymbol{S} = (S_1, \dots, S_{m+n}), -i'_k = Q^k_{n_k} - i_{n+k} \text{ for } k = 1, \dots, m,$$

and

$$-\boldsymbol{i}' = (-i'_1, \dots, -i'_m, -i_{m+1}, \dots, -i_{m+n}), \boldsymbol{\rho}^{-\boldsymbol{i}'} = \rho_1^{-i'_1} \cdots \rho_n^{-i'_m} \cdot \rho_{m+1}^{-i_{n+1}} \cdots \rho_{m+n}^{-i_{m+n}}.$$

The **enhanced Euler characteristic** of $\underline{\mathrm{SHG}}(-M, -\gamma)$ is

$$\begin{aligned}\chi_{\mathrm{en}}(\underline{\mathrm{SHG}}(-M,-\gamma)) &= j_*(\chi(\mathcal{SHG}_T(-M,-\gamma))) \\ &:= j_*\Big(\sum_{\substack{i_1\in[1,q_1],\dots,i_m\in[1,q_m]\\(i_{m+1},\dots,i_{m+n})\in\mathbb{Z}^n}} \chi(\underline{\mathrm{SHG}}(-M_T, -\gamma_T, \boldsymbol{S}, -\boldsymbol{i}')) \cdot \boldsymbol{\rho}^{-\boldsymbol{i}'}\Big) \in \mathbb{Z}[H]/\pm H.\end{aligned}$$

For $h \in H_1(M)$, let $\underline{\mathrm{SHG}}(-M, -\gamma, h)$ be image of the summand of $\mathcal{SHG}_T(-M, -\gamma)$ under the isomorphism in Propsition 4.2.18 whose corresponding element in $\chi_{\mathrm{en}}(\underline{\mathrm{SHG}}(-M, -\gamma))$ is h.

Remark 4.2.20. As mentioned in Remark 4.2.17, the definition of $\underline{\mathrm{SHG}}(-M, -\gamma, h)$ is not canonical, *i.e.* it may depend on many auxiliary choices. After fixing these choices, it is still only well-defined up to a global grading shift by multiplication by an element in $h_0 \in H_1(M)$. However, by Theorem 4.2.21, the enhanced Euler characteristic $\chi_{\mathrm{en}}(\underline{\mathrm{SHG}}(-M, -\gamma))$ only depends on (M, γ).

4.2.3 Identifying enhanced Euler characteristics

In this subsection, we prove Theorem 1.2.1.

Theorem 4.2.21 (Theorem 1.2.1)**.** *Suppose (M, γ) is a balanced sutured manifold and suppose $H = H_1(M)$. Then we have*

$$\chi_{\mathrm{en}}(\underline{\mathrm{SHI}}(-M, -\gamma)) = \chi(SFH(-M, -\gamma)) \in \mathbb{Z}[H]/\pm H.$$

Proof. First, we consider the case that (M, γ) is strongly balanced. By discussion in Subsection A.2.2, we can construct a $\mathbb{Z}$-grading on SFH associated to an admissible surface

$S \subset (M,\gamma)$. Hence we can apply the construction in previous subsection to SFH. We write $\mathcal{SHI}_T(M,\gamma)$ and $\mathcal{SFH}_T(M,\gamma)$ for the decompositions about $\underline{\mathrm{SHI}}(M,\gamma)$ and $SFH(M,\gamma)$ in Definition 4.2.16, respectively. By Proposition 4.2.18, we have

$$\mathcal{SHI}_T(-M,-\gamma) \cong \underline{\mathrm{SHI}}(-M,-\gamma). \tag{4.2.9}$$

$$\mathcal{SFH}_T(-M,-\gamma) \cong SFH(-M,-\gamma). \tag{4.2.10}$$

Moreover, by the proofs of Lemma 3.1.8 and Proposition 4.2.10, the isomorphism in (4.2.10) is induced by contact 2-handle attachments along meridians of tangle components of T. Hence by Lemma A.2.19, the isomorphism in (4.2.10) respects spin^c structures. This implies that there exists $\mathfrak{s}_0 \in \mathrm{Spin}^c(-M,-\gamma)$, such that for any $h \in H_1(M)$, the summand of $\mathcal{SFH}_T(-M,-\gamma)$ corresponding to h is isomorphic to $SFH(-M,-\gamma,s_0+h)$. In particular, we have

$$\chi_{\mathrm{en}}(SFH(-M,-\gamma)) := j_*(\chi(\mathcal{SFH}_T(-M,-\gamma)) = \chi(SFH(-M,-\gamma)) \in \mathbb{Z}[H]/\pm H,$$

where $j_* : \mathbb{Z}[H_1(M_T)] \to \mathbb{Z}[H_1(M)]$.

By definition, the spaces $\mathcal{SFH}_T(-M,-\gamma)$ and $\mathcal{SHI}_T(-M,-\gamma)$ are direct summands of $SFH(-M_T,-\Gamma)$ and $\underline{\mathrm{SHI}}(-M_T,-\Gamma)$ for some $\Gamma \subset \partial M_T$, respectively. By Lemma 4.2.13, the group $H_1(M_T)$ has no torsion. Hence by Theorem 4.1.7 and facts (4.1.1), (4.1.2), we have

$$\begin{aligned}\chi_{\mathrm{gr}}(\mathcal{SHI}_T(-M,-\gamma)) =& \chi_{\mathrm{gr}}(\mathcal{SFH}_T(-M,-\gamma)) \\ =& \chi(\mathcal{SFH}_T(-M,-\gamma)) \in \mathbb{Z}[H_1(M_T)]/\pm H_1(M_T).\end{aligned}$$

Thus, we have

$$\begin{aligned}\chi_{\mathrm{en}}(\underline{\mathrm{SHI}}(-M,-\gamma)) =& j_*(\chi_{\mathrm{gr}}(\mathcal{SHI}_T(-M,-\gamma))) \\ =& \chi_{\mathrm{en}}(SFH(-M,-\gamma)) \\ =& \chi(SFH(-M,-\gamma)) \in \mathbb{Z}[H]/\pm H\end{aligned}$$

Then we consider the case that (M,γ) is not strongly balanced. As mentioned in Remark A.2.7. If ∂M is not connected, we can construct a sutured manifold (M',γ') with connected boundary by attaching contact 1-handles (*c.f.* [Juh08, Remark 3.6]). The product disks in (M',γ') corresponding to these 1-handles are admissible surfaces, and only one summand in the associated $\mathbb{Z}$-grading is nontrivial. Hence there is a canonical way to consider $\chi_{\mathrm{en}}(\underline{\mathrm{SHG}}(-M',-\gamma'))$ as an element in $\mathbb{Z}[H_1(M)]/\pm H_1(M)$. We can consider $(-M',-\gamma')$

instead, and the above arguments about strongly balanced sutured manifolds apply to this case. □

4.3 Constrained knots in lens spaces

4.3.1 Preliminaries on 2-bridge links

In this subsection, we review some facts about 2-bridge links from [Ras02, BZ03, Mur08].

Definition 4.3.1. Suppose h is the height function given by the z-coordinate in $\mathbb{R}^3 \subset S^3$. A knot or a link in S^3 is called a **2-bridge knot** or a **2-bridge link** if it can be isotoped in a presentation so that h has two maxima and two minima on it. Such a presentation is called the **standard presentation** of the knot.

A 2-bridge link has two components. Each component is equivalent to the unknot. Suppose integers a and b satisfying $\gcd(a,b) = 1$ and $a > 1$. For every oriented lens space $L(a,b)$, there is a unique 2-bridge knot or link whose branched double cover space is diffeomorphic to $L(a,b)$. Let $\mathfrak{b}(a,b)$ denote the knot or link related to $L(a,b)$. It is a knot if a is odd, and a link if a is even. Thus, the classification of 2-bridge knots or links depends on the classification of lens spaces [Bro60]. For $i = 1,2$, two 2-bridge knots or links $\mathfrak{b}(a_i,b_i)$ are equivalent if and only if $a_1 = a_2 = a$ and $b_1 \equiv b_2^{\pm 1} \pmod{a}$.

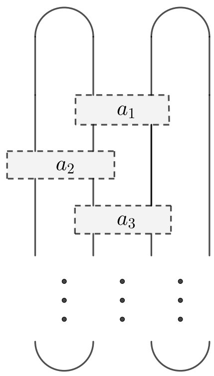


Figure 4.4 2-bridge.

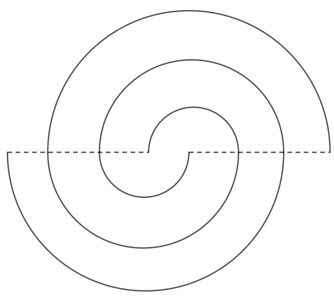


Figure 4.5 $\mathfrak{b}(3,1)$.

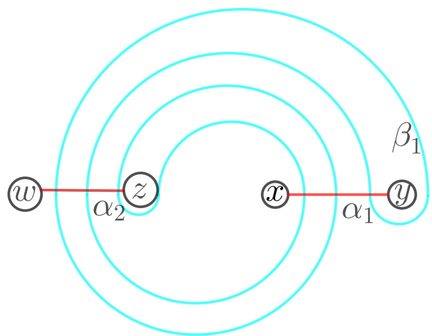


Figure 4.6 Diagram of $E(\mathfrak{b}(3,1))$.

Suppose a/b is represented as the continued fraction

$$[0; a_1, -a_2, \ldots, (-1)^{m+1} a_m] = 0 + \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots}}}.$$

Moreover, suppose m is odd. The standard presentation of a 2-bridge knot or link $\mathfrak{b}(a,b)$ looks like Figure 4.4, where $|a_i|$ for $i \in [1,m]$ represent numbers of half-twists in the boxes and signs of a_i represent signs of half-twists. Different choices of continued fractions give the same knot or link. For any 2-bridge knot or link, the numbers $(-1)^{i+1}a_i$ can be all positive, which implies any 2-bridge knot or link is alternating.

The knot or link $\mathfrak{b}(a,b)$ admits another canonical presentation known as the **Schubert normal form**. It induces a Heegaard diagram of $E(\mathfrak{b}(a,b))$ and a doubly-pointed Heegaard diagram of $\mathfrak{b}(a,b)$. Figure 4.5 gives an example of the Schubert normal form of $\mathfrak{b}(3,1)$ and Figure 4.6 is the corresponding Heegaard diagram of the knot complement. The corresponding doubly-pointed Heegaard diagram is obtained by replacing α_2 by two basepoints z and w. Two horizontal strands in the Schubert normal form are arcs near two maxima in the standard presentation. Thus, two 1-handles attached to points w,z and x,y in Figure 4.6 are neighborhoods of these arcs, respectively.

Proposition 4.3.2 ([Ras02])**.** *Suppose $K = \mathfrak{b}(a,b)$ with b odd and $|b| < a$. The symmetrized Alexander polynomial $\Delta_K(t)$ and the signature $\sigma(K)$ satisfy*

$$\Delta_K(t) = t^{-\frac{\sigma(K)}{2}} \sum_{i=0}^{a-1} (-1)^i t^{\sum_{j=0}^{i}(-1)^{\lfloor \frac{ib}{a} \rfloor}}, \quad \sigma(K) = \sum_{i=1}^{a-1} (-1)^{\lfloor \frac{ib}{a} \rfloor}.$$

4.3.2 Parameterization

For a constrained knot K, there is a standard diagram $(T^2, \alpha_1, \beta_1, z, w)$ of K defined in the end of Section 1.2. Based on standard diagrams, we describe the parameterization of constrained knots. For integers p,q,q' satisfying

$$\gcd(p,q) = \gcd(p,q') = 1 \text{ and } qq' \equiv 1 \pmod{p},$$

we know that $L(p,q)$ is diffeomorphic to $L(p,q')$ [Bro60]. Suppose (T^2, α_0, β_0) is the standard diagram of $L(p,q')$, *i.e.*, the curve β_0 is obtained from a straight line of slope p/q' in $\mathbb{R}^2$, and suppose that the diagram $(T^2, \alpha_1, \beta_1, z, w)$ is induced by (T^2, α_0, β_0) as in Section 1.2. The curves α_0 and β_0 divide T^2 into p regions, which are parallelograms in Figure 1.1; see also the left subfigure of Figure 4.7. A new diagram C is obtained by gluing top edges and bottom edges of parallelograms. We can shape C into a square. An example is shown in Figure 4.7, where $p = 5, q = 3, q' = 2$.

For $i \in \mathbb{Z}/p\mathbb{Z}$, let D_i denote rectangles in C, ordered from the bottom edge to the top edge. Since $qq' \equiv 1 \pmod{1}$ and we start with the standard diagram of $L(p,q')$, we know that the right edge of D_j is glued to the left edge of D_{j+q}. The bottom edge e_b of D_1 is glued

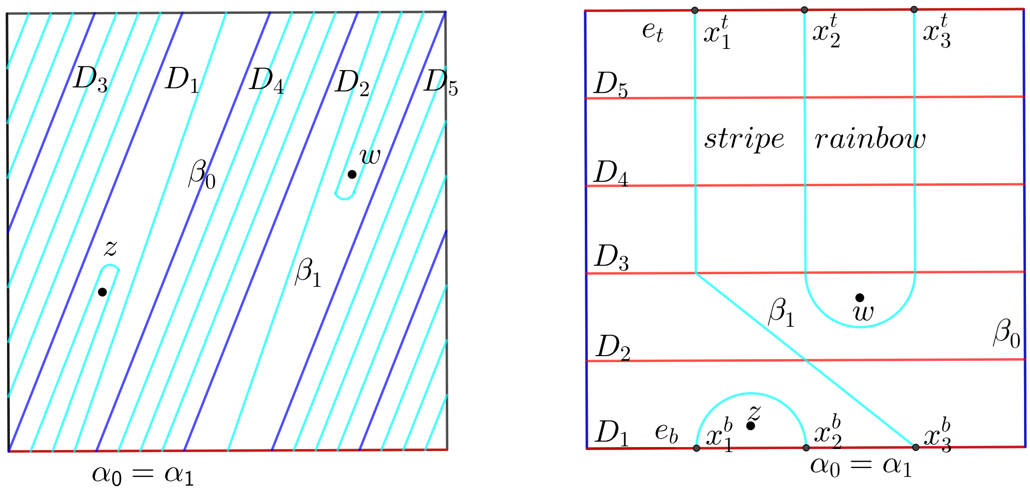


Figure 4.7 Heegaard diagrams of $C(5,3,2,3,1)$.

to the top edge e_t of D_p. By definition of a constrained knot, the curve α_1 is the same as α_0 and the curve β_1 is disjoint from β_0. Thus, in this new diagram C, the curve α_1 is the union of p horizontal lines and β_1 is the union of strands which are disjoint from vertical edges of D_i for $i \in \mathbb{Z}/p\mathbb{Z}$.

Similar to the definitions for $(1,1)$ knots, strands in the standard diagram of a constrained knot are called **rainbows** and **stripes**. Boundary points of a rainbow and a stripe are called **rainbow points** and **stripe points**, respectively. A rainbow must bound a basepoint, otherwise it can be removed by isotopy. Numbers of rainbows on e_b and e_t are the same since the numbers of rainbow points are the same. Without loss of generality, suppose z is in all rainbows on e_b and w is in all rainbows on e_t. Let x_i^b and x_i^t for $i \in [1,u]$ be boundary points on the bottom edge and the top edge, respectively, ordered from left to right in the right subfigure of Figure 4.7.

Lemma 4.3.3. *The number u of boundary points on e_b or e_t is odd. When $u = 1$, there is no rainbow and only one stripe. When $u > 1$, there exists an integer $v \in (0, u/2)$ so that one of the following cases happens:*

(i) the set $\{x_i^b | i \le 2v\} \cup \{x_i^t | i > u - 2v\}$ contains all rainbow points;

(ii) the set $\{x_i^t | i \le 2v\} \cup \{x_i^b | i > u - 2v\}$ contains all rainbow points.

Proof. The algebraic intersection number of β_1 and e_b is odd. Hence u is also odd. If $u = 1$, then the argument is clear.

Suppose $u > 1$, we show the last argument in three steps. Firstly, if both x_i^b and x_j^b are boundary points of the same rainbow R, then x_k^b for $i < k < j$ are all rainbow points, otherwise the stripe corresponding to the stripe point x^b_k would intersect R. Thus, rainbow points on e_b are consecutive. The same assertion holds for x^t_i.

Secondly, one of x^b_1 and x^t_1 must be a rainbow point. Indeed, if this were not true, then both x^b_1 and x^t_1 would be stripe points. They cannot be boundary points of the same stripe, otherwise β_1 would not be connected. They cannot be boundary points of different stripes, otherwise two corresponding stripes would intersect each other. Thus, the assumption is false. Similarly, one of x^b_u and x^t_u must be a rainbow point.

Finally, if x^b_1 is a rainbow point, then x^b_u cannot be a rainbow point, otherwise all points were rainbow points. As discussed above, the point x^t_u is a rainbow point. Since the number of rainbow points on e_t is even, there exists an integer v satisfying Case (i). If x^t_1 is a rainbow point, similar argument implies there exists v satisfying Case (ii). □

When $u = 1$, after isotoping β_1, suppose the unique stripe is a vertical line in $C - \{z, w\}$. By moving z through the left edge or the right edge if necessary, suppose basepoints z and w are in different sides of the stripe. If z is on the left of the stripe, set $v = 0$. If z is on the right of the stripe, set $v = 1$.

Then suppose $u > 1$. When in Case (i) of Lemma 4.3.3, rainbows on e_b connect x^b_i to x^b_{2v+1-i} for $i \in [1, v]$, rainbows on e_t connect x^t_{u+1-i} to x^t_{u-2v+i} for $i \in [1, v]$, and stripes connect x^b_j to x^t_{u+1-j} for $j \in [2v+1, u]$. When in Case (ii) of Lemma 4.3.3, the setting is obtained by replacing i and j by $u+1-i$ and $u+1-j$, respectively. Without loss of generality, suppose z is in D_1, and w is in D_l. Note that now basepoints cannot be moved through vertical edges of C. Otherwise the rainbows would intersect the vertical edges, which contradicts the definition of the constrained knot. Then we parameterize constrained knots in $L(p, q')$ by the tuple (l, u, v) for Case (i) and $(l, u, u-v)$ for Case (ii). Since β_1 is connected, we have $\gcd(u, v) = 1$. In summary, the following theorem holds.

Theorem 4.3.4. *Constrained knots are parameterized by five integers* (p, q, l, u, v)*, where* $p > 0, q \in [1, p-1], l \in [1, p], u > 0, v \in [0, u-1]$*, u is odd, and* $\gcd(p, q) = \gcd(u, v) = 1$*. Moreover,* $v \in [1, u-1]$ *when* $u > 1$ *and* $v \in \{0, 1\}$ *when* $u = 1$*.*

Note that the parameter v in Theorem 4.3.4 is different from the integer v in Case (ii) of Lemma 4.3.3. Intuitively, for $v \in [1, u-1]$ in the parameterization (p, q, l, u, v) with $u > 1$, the number $\min\{v, u-v\}$ is the number of rainbows around a basepoint.

For paramters (p, q, l, u, v), let $C(p, q, l, u, v)$ denote the corresponding constrained knot. When considering the orientation, let $C(p, q, l, u, v)^+$ denote the knot induced by $(T, \alpha_1, \beta_1, z, w)$ and let $C(p, q, l, u, v)^-$ denote the knot induced by $(T, \alpha_1, \beta_1, w, z)$. For $q \neq [1, p-1]$ and $l \neq [1, p]$, consider the integers q and l modulo p. If $u > 1$ and $v \neq [1, u-1]$, consider the integer v modulo u. For $p < 0$, let $C(p, q, l, u, v)$ denote $C(-p, -q, l, u, v)$.

Remark 4.3.5. The knot $C(p,q,l,u,v)$ is in $L(p,q')$, where $qq' \equiv 1 \pmod p$. Though $L(p,q)$ is diffeomorphic to $L(p,q')$, constrained knots $C(p,q,l,u,v)$ and $C(p,q',l,u,v)$ is not necessarily equivalent. For example, constrained knots $C(5,2,3,3,1)$ and $C(5,3,3,3,1)$ are not equivalent.

Then we provide some basic propositions of constrained knots.

Proposition 4.3.6. *$C(p,-q,l,u,-v)$ is the mirror image of $C(p,q,l,u,v)$ for $u>1$. $C(p,-q,l,1,1)$ is the mirror image of $C(p,q,l,1,0)$.*

Proof. It follows from the vertical reflection of the standard diagram. □

Hence we only consider $C(p,q,l,u,v)$ with $0 \le 2v < u$ in the rest of this section.

Proposition 4.3.7. $C(1,0,1,u,v) \cong \mathfrak{b}(u,v)$.

Proof. By cutting along α_1 and a small circle around x in Figure 4.6, the doubly-pointed diagram of a 2-bridge knot can be shaped into a square. This proposition is clear by comparing this diagram with the new diagram C related to $C(1,0,1,u,v)$. □

Proposition 4.3.8. *For any fixed orientations of α_1 and β_1 in the standard diagram of a constrained knot, intersection points x_i^b have alternating signs and adjacent strands of β_1 in the new diagram C have opposite orientations.*

Proof. From a similar observation in the proof of Proposition 4.3.7, for $C(p,q,l,u,v)$, the curve β_1 in the new diagram C is same as the curve β in the doubly-pointed Heegaard diagram of $\mathfrak{b}(u,v)$. Thus, it suffices to consider the 2-bridge knot $\mathfrak{b}(u,v)$. The Schubert normal form of $\mathfrak{b}(u,v)$ is the union of two dotted horizontal arcs behind the plane and two winding arcs on the plane. Suppose γ is one of the winding arc. Then $\beta_1 = \partial N(\gamma)$ cuts the plane into two regions, the inside region $\mathrm{int}N(\gamma)$ and the outside region $\mathbb{R}^2 - N(\gamma)$. Points x and y in Figure 4.6 are in different regions and points x_i^b are on the arc connecting x to y. Since regions on different sides of β_1 must be different, the arc connecting x to y is cut by x_i^b into pieces that lie in the inside region and the outside region alternately. For each piece of the arc, the endpoints are boundary points of a connected arc in β_1. Thus, signs of x_i^b are alternating. The orientations on strands of β_1 are induced by signs of x_i^b. Hence adjacent strands of β_1 have opposite orientations. □

Proposition 4.3.9. *For $K = C(p,q,l,1,0)$, we have a presentation of the homology*

$$H_1(E(K)) \cong \langle [a],[m] \rangle / (p[a]+k[m]) \cong \mathbb{Z} \oplus \mathbb{Z}/\gcd(p,k)\mathbb{Z},$$

where m is the meridian as in Figure 4.8, a is the core curve of α_0-handle and $k \in (0,p]$ satisfies $k-1 \equiv (l-1)q^{-1} \pmod p$.

Proof. This follows from [Ras07, Section 3.3]. □

4.3.3 Knot Floer homology

Throughout this section, suppose $K = C(p,q,l,u,v)$ is a constrained knot in $Y = L(p,q')$, where $qq' \equiv 1 \pmod{p}$. Write $H_1 = H_1(E(K))$ and $\widehat{HFK}(K) = \widehat{HFK}(Y,K)$ for short. For any homogeneous element $x \in \widehat{HFK}(K)$, let $\mathrm{gr}(x) \in H_1$ be the Alexander grading of x. Note that the Alexander grading is well-defined up to a global grading shift, *i.e.* up to multiplication by an element in H_1. However, the difference $\mathrm{gr}(x) - \mathrm{gr}(y)$ for two homogeneous elements x and y is always well-defined. This difference can be calculated explicitly by the doubly-pointed Heegaard diagram of the knot by the approach in [Ras07, Section 3.3].

For a constrained knot K, we will show $\widehat{HFK}(K)$ totally depends on $\chi(\widehat{HFK}(K))$. Explicitly this means that, for any $\mathfrak{s} \in \mathrm{Spin}^c(Y,K)$, the dimension of $\widehat{HFK}(K,\mathfrak{s})$ is the same as the absolute value $|\chi(\widehat{HFK}(K,\mathfrak{s})|$. Then by Friedl-Juhász-Rasmussen [FJR09], we know $\widehat{HFK}(K,\mathfrak{s})$ is determined by the Turaev torsion of $E(K)$.

As shown in Figure 4.7 and Figure 4.8, suppose e^j is the top edge of D_j and x_i^j is the intersection point of e^j and β_1 for $j \in \mathbb{Z}/p\mathbb{Z}, i \in [1,u(j)]$. Let $x^j_{\mathrm{middle}} = x^j_{(u(j)+1)/2}$ be middle points. It is clear that $\mathfrak{s}_z(x^{j_1}_{i_1}) = \mathfrak{s}_z(x^{j_2}_{i_2})$ if and only if $j_1 = j_2$. For any integer $j \in [1,p]$, define $\mathfrak{s}_j = \mathfrak{s}_z(x^j_{middle}) \in \mathrm{Spin}^c(Y)$.

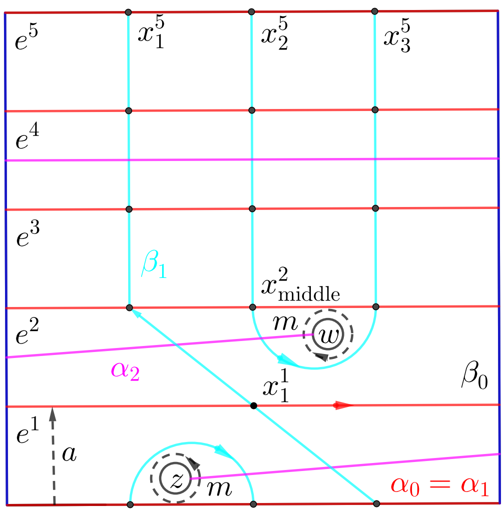


Figure 4.8 Heegaard diagram of $E(C(5,3,2,3,1))$.

Lemma 4.3.10. *For $K = C(p,q,l,u,v)$ with $u > 2v > 0$, suppose $k \in (0,p]$ is the integer satisfying $k-1 \equiv (l-1)q^{-1} \pmod p$. Define*

$$k' = \begin{cases} k-2 & v \text{ odd,} \\ k & v \text{ even.} \end{cases}$$

Suppose $d = \gcd(p,k')$. Then there is a presentation of the homology H_1:

$$H_1 = H_1(E(K)) \cong \langle [a],[m] \rangle / (p[a] + k'[m]) \cong \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z},$$

where m is the circle in Figure 4.8 and a is the core curve of α_0-handle.

Proof. Suppose β_1 is oriented so that the orientation of the middle stripe is from bottom to top. Let $[\beta_1(p,q,l,u,v)]$ denote the homology class of β_1 corresponding to $C(p,q,l,u,v)$. By Proposition 4.3.8, orientations of rainbows around a basepoint are alternating. Note that moving all rainbows of β_1 across basepoints gives the diagram of $C(p,q,l,1,0)$. Then

$$\begin{cases} [\beta_1(p,q,l,u,v)] + 2[m] = [\beta_1(p,q,l,1,0)] & v \text{ odd,} \\ [\beta_1(p,q,l,u,v)] = [\beta_1(p,q,l,1,0)] & v \text{ even.} \end{cases}$$

Then this proposition follows from Proposition 4.3.9. Note that $[a]$ and $[m]$ correspond to core curves of α_1 and α_2 and the relation in the presentation of H_1 corresponds to algebraic intersection numbers $\alpha_1 \cdot \beta$ and $\alpha_2 \cdot \beta$. □

Lemma 4.3.11. *For $K = C(p,q,l,u,v)$ with $u > 2v \geq 0$, suppose H_1 is presented as in Lemma 4.3.10. For any integer $j \in [1,p]$, let $\mathfrak{s}_j = \mathfrak{s}_z(x^j_{middle})$ for intersection points x^j_{middle} in Figure 4.8. Then for any j, the group $\widehat{HFK}(K,\mathfrak{s}_j)$ is determined by its Euler characteristic.*

Moreover, suppose integers u' and v' satisfy $u' = u - 2v$ and $v' \equiv v \pmod{u'}$. Let $\Delta_1(t)$ and $\Delta_2(t)$ be Alexander polynomials of $\mathfrak{b}(u,v)$ and $\mathfrak{b}(u',v')$, respectively. Then

$$\chi(\widehat{HFK}(K,\mathfrak{s}_j)) = \begin{cases} \Delta_1([m]) & j \in [l,p], \\ \Delta_2([m]) & j \in [1,l-1], \end{cases}$$

as elements in $\mathbb{Z}[H_1]/\pm H_1$.

Proof. For $j \in [1,p]$, consider the edge e^j and the intersection numbers x^j_i of e^j and β_1 in the diagram C. Suppose $(e^j)'$ is the curve obtained by identifying two endpoints of e^j. For $j \in [l,p]$, the diagram $(T^2,(e^j)',\beta_1,z,w)$ is the same as the diagram of $K_1 = \mathfrak{b}(u,v)$.

For $j \in [1, l-1]$, we claim that the diagram $(T^2, (e^j)', \beta_1, z, w)$ is isotopic to the diagram of $K_2 = \mathfrak{b}(u', v')$.

The fact that $u' = u - 2v$ follows directly from the number of intersection points of $(e^j)'$ and β_1, which is the same as the number of stripes. Then we consdier v'. Let $D = N(x^p_{middle})$ be the neighborhood of x^p_{middle} so that D contains all rainbows. Consider the isotopy obtained by rotating D counterclockwise. If $v > u'$, after rotation, the resulting diagram has $v - u'$ rainbows. The formula for v' follows by induction.

2-bridge knots are alternating, hence are thin [OS03]. By comparing the number of generators of $\widehat{CFK}(K_i)$ for $i = 1, 2$ from $(T^2, (e^j)', \beta_1, z, w)$ and the dimension of $\widehat{HFK}(K_i)$ from the Alexander polynomial (*c.f.* Proposition 4.3.2), we know there is no differential on $\widehat{CFK}(K_i)$. This fact can also be shown by a direct calculation following the method in [GMM05]. Thus, the constrained knot K is also thin (in the similar sense to the thinness for knots in S^3) and there is no differential on $\widehat{CFK}(K, \mathfrak{s}_j)$. In particular, the group $\widehat{CFK}(K, \mathfrak{s}_j)$ is determined by its Euler characteristic.

Similar to the proof of [Ras02, Lemma 3.4], for $j \in [l, p]$, we have

$$\mathrm{gr}(x^j_{i+1}) - \mathrm{gr}(x^j_i) = [m]^{(-1)^{\lfloor \frac{iv}{u} \rfloor}}.$$

For $j \in [1, l-1]$, just replace u and v by u' and v' in the above formula, respectively. Comparing the formula of the Alexander polynomial in Proposition 4.3.2, we conclude the formula of $\chi(\widehat{HFK}(K, \mathfrak{s}_j))$. □

Lemma 4.3.12. *Consider integers k, k' and the presentation of H_1 as in Lemma 4.3.10.*

For $j \neq 0, l-1$, $$\mathrm{gr}(x^{j+1}_{middle}) - \mathrm{gr}(x^j_{middle}) = \begin{cases} [a] + [m] & \text{if } jq^{-1} \equiv 1, \dots, k-2 \pmod p \\ [a] & \text{otherwise.} \end{cases}$$

For $l \neq 1$ and $j = 0, l-1$, $$\mathrm{gr}(x^{j+1}_{middle}) - \mathrm{gr}(x^j_{middle}) = \begin{cases} [a] + [m] & v \text{ even} \\ [a] & v \text{ odd.} \end{cases}$$

For $l = 1$, $$\mathrm{gr}(x^{j+1}_{middle}) - \mathrm{gr}(x^j_{middle}) = \begin{cases} [a] + [m] & v \text{ even} \\ [a] - [m] & v \text{ odd.} \end{cases}$$

Proof. For $j = 0, l-1$, the constrained knot is a simple knot in the sense of [Ras07]. Then the proof is based on Fox calculus (*c.f.* [Ras07, Proposition 6.1]). For a general constrained knot and $j \neq 0, l-1$, the proof in [Ras07] still works because orientations of strands are alternating. The differences of gradings for $j = 0$ and $j = l-1$ are the same because z and w

are symmetric by rotation. The proof follows from the following equations

$$\sum_{j=0}^{p-1} \mathrm{gr}(x^{j+1}_{\text{middle}}) - \mathrm{gr}(x^{j}_{\text{middle}}) = 0 \in H_1 \text{ and } p[a] + k'[m] = 0 \in H_1.$$

□

Corollary 4.3.13. *Suppose $K = C(p,q,l,u,v)$ is a constrained knot in $Y = L(p,q')$, where $qq' \equiv 1 \pmod p$. For any integer $j \in [1,p]$, let $\mathfrak{s}_j = \mathfrak{s}_z(x^j_{middle}) \in \mathrm{Spin}^c(Y)$ for intersection points x^j_{middle} in Figure 4.8. Then $\mathfrak{s}_{j+1} - \mathfrak{s}_j$ only depends on p and q.*

Proof. By the map $H_1(E(K))/([m]) \to H_1(Y)$, the grading difference $\mathrm{gr}(x^{j+1}_{\text{middle}}) - \mathrm{gr}(x^{j}_{\text{middle}})$ is mapped to $\mathfrak{s}_{j+1} - \mathfrak{s}_j$, which only depends on the image of $[a]$. □

Lemma 4.3.14. *Consider $\mathfrak{b}(u,v)$ and $\mathfrak{b}(u',v')$ as in Lemma 4.3.11. Then*

$$\sigma(\mathfrak{b}(u',v')) = \begin{cases} \sigma(\mathfrak{b}(u,v)) & v \text{ even}, \\ \sigma(\mathfrak{b}(u,v)) + 2 & v \text{ odd}. \end{cases}$$

Proof. Consider standard presentations of 2-bridge knots in Subsection 4.3.1. It is easy to see $\mathfrak{b}(u,v)$ and $\mathfrak{b}(u',v')$ form two knots in the skein relation. By the skein relation formula of signatures of knots, we can conclude this lemma. Moreover, we provide another proof based on the Alexander grading as follows.

By the algorithm of the Alexander grading, we have

$$\mathrm{gr}(x^1_{u'}) - \mathrm{gr}(x^0_u) = [a] + [m].$$

From the rotation symmetry and the formula of the signature in Proposition 4.3.2,

$$\mathrm{gr}(x^0_u) - \mathrm{gr}(x^0_{\text{middle}}) = \mathrm{gr}(x^0_{\text{middle}}) - \mathrm{gr}(x^0_1) = \frac{\sigma(\mathfrak{b}(u,v))}{2}[m],$$

$$\mathrm{gr}(x^1_{u'}) - \mathrm{gr}(x^1_{\text{middle}}) = \mathrm{gr}(x^1_{\text{middle}}) - \mathrm{gr}(x^1_1) = \frac{\sigma(\mathfrak{b}(u',v'))}{2}[m].$$

Then this lemma follows from these equations and Lemma 4.3.12. □

Theorem 4.3.15. *For a constrained knot $K = C(p,q,l,u,v)$, consider the Alexander polynomials $\Delta_1(t)$ and $\Delta_2(t)$ in Lemma 4.3.11. Then $\widehat{HFK}(K)$ with Alexander grading and Mod 2 Maslov grading is determined by its Euler characteristic, which is calculated by the following* *formula:*

$$\chi(\widehat{HFK}(K)) = \Delta_1([m])\sum_{j=l}^{p} \mathrm{gr}(x^j_{middle}) + \Delta_2([m])\sum_{j=1}^{l-1} \mathrm{gr}(x^j_{middle}) \tag{4.3.1}$$

Proof. By the result of Lemma 4.3.11, we only need to consider the (relative) signs of intersection points corresponding to different spin^c structures. By Proposition 4.3.8, signs of intersection points x^j_i for fixed j are alternating. Since u and $u' = u - 2v$ are odd, signs of x^j_1 and $x^j_{u(j)}$ are the same, where $u(j)$ is either u or u' by Lemma 4.3.11. From the diagram, signs of $x^j_{u(j)}$ for $j \in [0, l]$ are the same and signs of x^k_1 for $k \in [l, p]$ are the same. Thus, we obtain Formula (4.3.1). □

All terms in Formula 4.3.1 can be calculated by Lemma 4.3.12 and Lemma 4.3.14. Thus, we obtain an algorithm of $\widehat{HFK}(K)$ for a constrained knot K.

Chapter 5

Calculation by Dehn surgery formulae

In this Chapter, we focus on balanced sutured manifolds that are obtained from knots and closed 3-manifolds and study the relation between instanton knot homology $\underline{\mathrm{KHI}}$ and framed instanton homology $I^\sharp(Y)$ (Definition 2.3.17).

In the first section, we constructed differentials d_+ and d_- on instanton knot homology $\underline{\mathrm{KHI}}(Y,K)$ for a rationally null-homologous knot K in a closed 3-manifold Y and prove the large surgery formula (Theorem 1.3.10). The proof is purely algebraic. The main ingredient is the octahedral axiom in Subsection 2.2.3.

In the second section, we prove some vanishing results about contact elements and contact gluing maps, which are of independent interest for contact geometry.

In the third section, we use results in former sections to prove a generalization of Theorem 1.3.6. Many ideas come from the proof [OS05b, Theorem 1.2] in Heegaard Floer theory due to Ozsváth-Szabó.

5.1 Differentials and the large surgery formula

5.1.1 The caonical basis on the torus boundary

In this subsection, we provide a canonical way to fix the basis on the boundary of the knot complement and introduce some notations about sutures.

Suppose Y is a closed 3-manifold and $K \subset Y$ is a null-homologous knot. Let $Y\backslash K$ be the knot complement $Y\backslash\mathrm{int}(N(K))$. Any Seifert surface S of K gives rise to a framing on $\partial Y\backslash K$: the longitude λ can be picked as $S\cap\partial Y\backslash K$ with the induced orientation from S, and the meridian μ can be picked as the meridian of the solid torus $N(K)$ with the orientation so that $\mu\cdot\lambda=-1$. The 'half lives and half dies' fact for 3-manifolds implies that the following map has a 1-dimensional image:

$$\partial_* : H_2(Y\backslash K, \partial Y\backslash K; \mathbb{Q}) \to H_1(\partial Y\backslash K; \mathbb{Q}).$$

Hence any two Seifert surfaces lead to the same framing on $\partial Y\backslash K$.

Definition 5.1.1. The framing (μ, λ) defined as above is called the **canonical framing** of (Y, K). With respect to this canonical framing, let

$$\widehat{Y}_{q/p} = Y\backslash K \cup_\phi S^1 \times D^2$$

be the 3-manifold obtained from Y by a q/p surgery along K, *i.e.*,

$$\phi(\{1\} \times \partial D^2) = q\mu + p\lambda.$$

We also write $\widehat{Y}_\alpha$ for $\widehat{Y}_{q/p}$, where $\alpha = \phi(\{1\} \times \partial D^2)$. When the surgery slope is understood, we also write $\widehat{Y}_{q/p}$ simply as $\widehat{Y}$. Let $\widehat{K}$ be the dual knot, *i.e.*, the image of $S^1 \times \{0\} \subset S^1 \times D^2$ in $\widehat{Y}$ under the gluing map.

Convention. Throughout this section, we will always assume that $\gcd(p, q) = 1$ and $q > 0$ or $(p, q) = (1, 0)$ for a Dehn surgery. Especially, the original pair (Y, K) can be thought of as a pair $(\widehat{Y}, \widehat{K})$ obtained from (Y, K) by the $1/0$ surgery. Moreover, we will always assume that the knot complement $Y\backslash K$ is irreducible. This is because if $Y\backslash K$ is not irreducible, then $Y\backslash K \cong Y'\backslash K' \sharp Y''$ for some closed 3-manifold Y', Y'' and a null-homologous knot $K' \subset Y'$. By the connected sum formula [Li20, Section 1.8], we have

$$\underline{\mathrm{SHI}}(Y\backslash K, \gamma) \cong \underline{\mathrm{SHI}}(Y'\backslash K', \gamma) \otimes I^\sharp(Y'')$$

for any suture γ. Hence all results hold after tensoring $I^\sharp(Y'')$.

Next, we describe various families of sutures on the knot complement. Suppose $K \subset Y$ is a null-homologous knot and the pair $(\widehat{Y}, \widehat{K})$ is obtained from (Y, K) by a q/p surgery. Note we can identify the complement of $K \subset Y$ with that of $\widehat{K} \subset \widehat{Y}$, *i.e.* $\widehat{Y}\backslash\widehat{K} = Y\backslash K$.

On $\partial Y\backslash K$, there are two framings: One comes from K, and we write longitude and meridian as λ and μ, respectively. The other comes from $\widehat{K}$. Note only the meridian $\hat{\mu}$ of $\widehat{K}$ is well-defined, and by definition, it is $\hat{\mu} = q\mu + p\lambda$.

Definition 5.1.2. If $p = 0$, then $q = 1$ and $\hat{\mu} = \mu$. We can take $\hat{\lambda} = \lambda$. If $(q, p) = (0, 1)$, then we take $\hat{\lambda} = -\mu$. If $p, q \neq 0$, then we take $\hat{\lambda} = q_0\mu + p_0\lambda$, where (q_0, p_0) is the unique pair of integers so that the following conditions are true.

(1) $0 \le |p_0| < |p|$ and $p_0 p \le 0$.

(2) $0 \le |q_0| < |q|$ and $q_0 q \le 0$.

(3) $p_0 q - p q_0 = 1$.

In particular, if $(q,p) = (n,1)$, then $\hat{\lambda} = -\mu$.

For a homology class $x\lambda + y\mu$, let $\gamma_{x\lambda+y\mu}$ be the suture consisting of two disjoint simple closed curves representing $\pm(x\lambda + y\mu)$ on $\partial Y\backslash K$. Furthermore, for $n \in \mathbb{Z}$, define

$$\widehat{\Gamma}_n(q/p) = \gamma_{\hat{\lambda}-n\hat{\mu}} = \gamma_{(p_0-np)\lambda+(q_0-nq)\mu}, \text{ and } \widehat{\Gamma}_\mu(q/p) = \gamma_{\hat{\mu}} = \gamma_{p\lambda+q\mu}.$$

Suppose $(q_n, p_n) \in \{\pm(q_0 - nq, p_0 - np)\}$ such that $q_n \ge 0$. Note that there might be a sign ambiguity of q_0: if $q > 0$, then by term (2) above $q_0 < 0$; but here $n = 0$ implies the new q_0 is the opposite number of the original q_0. We keep this ambiguity and uses the first definition of q_0 only for $\hat{\lambda}$ and uses the second definition only in the formula of q_n.

When emphasizing the choice of $\hat{\mu}$, we also write $\widehat{\Gamma}_n(\hat{\mu})$ and $\widehat{\Gamma}_\mu(\hat{\mu})$. When $\hat{\lambda}$ and $\hat{\mu}$ are understood, we omit the slope q/p and simply write $\widehat{\Gamma}_n$ and $\widehat{\Gamma}_\mu$. When $(q,p) = (1,0)$, we write Γ_n and Γ_μ instead.

Remark 5.1.3. Since the two components of the suture must be given opposite orientations, the notations $\gamma_{x\lambda+y\mu}$ and $\gamma_{-x\lambda-y\mu}$ represent the same suture on the knot complement $Y\backslash K$. Our choice makes $q_{n+1} \le q_n$ for $n < -1$ and $q_{n+1} \ge q_n$ for $n \ge 0$.

5.1.2 Bypasses on knot complements

Suppose Y is a closed 3-manifold and $K \subset Y$ is a null-homologous knot. Let (μ, λ) be the canonical framing on $Y\backslash K$ in Definition 5.1.1. Suppose y_3/x_3 is a surgery slope with $y_3 \ge 0$. According to Honda [Hon00, Section 4.3], there are two basic bypasses on the balanced sutured manifold $(Y\backslash K, \gamma_{(x_3,y_3)})$, whose arcs are depicted as in Figure 5.1. The sutures involved in the bypass triangles were described explicitly in Honda [Hon00, Section 4.4.4].

Definition 5.1.4. For a surgery slope y_3/x_3 with $y_3 \ge 0$, suppose its continued fraction is

$$\frac{y_3}{x_3} = [a_0, a_1, \ldots, a_n] = a_0 - \cfrac{1}{a_1 - \cfrac{1}{\cdots - \frac{1}{a_n}}},$$

where integers $a_i < -1$. If $y_3 > -x_3 > 0$, let

$$\frac{y_1}{x_1} = [a_0, \ldots, a_{n-1}] \text{ and } \frac{y_2}{x_2} = [a_0, \ldots, a_n + 1].$$

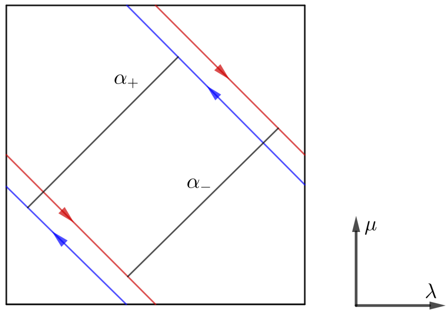


Figure 5.1 Bypass arcs on $\gamma_{(1,-1)}$.

When $a_i = -2$ for integer $i \in (k,n]$ and $a_k \neq -2$, we know

$$[a_0,\dots,a_n+1] = [a_0,\dots,a_k+1].$$

So we can always assume $a_n \neq -2$. If $-x_3 > y_3 > 0$, we do the same thing for $x_3/(-y_3)$. If $y_3 > x_3 > 0$, we do the same thing for $y_3/(-x_3)$. If $x_3 > y_3 > 0$, we do the same thing for $x_3/(-y_3)$. If $y_3/x_3 = 1/0$, then set $y_1/x_1 = 0/1$ and $y_2/x_2 = 1/(-1)$. If $y_3/x_3 = 0/1$, then set $y_1/x_1 = 1/(-1)$ and $y_1/x_1 = 0/1$. We always require that $y_1 \geq 0$ and $y_2 \geq 0$.

Remark 5.1.5. It is straightforward to use induction to verify that for $y_3 > -x_3 > 0$,

$$x_3 = x_1 + x_2 \text{ and } y_3 = y_1 + y_2.$$

The bypass exact triangle in Theorem 2.3.38 becomes the following.

Proposition 5.1.6. *Suppose $K \subset Y$ is a null-homologous knot, and suppose the surgery slopes y_i/x_i for $i \in \{1,2,3\}$ are defined as in Definition 5.1.4. Suppose the indices are considered mod 3. Let $\psi^{y_i/x_i}_{+,y_{i+1}/x_{i+1}}$ and $\psi^{y_i/x_i}_{-,y_{i+1}/x_{i+1}}$ be bypass maps from two different bypasses, respectively. Then there are two exact triangles related to $\psi^{y_i/x_i}_{+,y_{i+1}/x_{i+1}}$ and $\psi^{y_i/x_i}_{-,y_{i+1}/x_{i+1}}$, respectively.*

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(-Y\backslash K,-\gamma_{(x_2,y_2)}) & \xrightarrow{\psi^{y_2/x_2}_{\pm,y_3/x_3}} & \underline{\mathrm{SHI}}(-Y\backslash K,-\gamma_{(x_3,y_3)}) \\ {\scriptstyle \psi^{y_1/x_1}_{\pm,y_2/x_2}} \nwarrow & & \swarrow {\scriptstyle \psi^{y_3/x_3}_{\pm,y_1/x_1}} \\ & \underline{\mathrm{SHI}}(-Y\backslash K,-\gamma_{(x_1,y_1)}) & \end{array}$$

Proposition 5.1.7. *Suppose $K \subset Y$ is a null-homologous knot and suppose the pair $(\widehat{Y},\widehat{K})$ is obtained from (Y,K) by a q/p surgery. Suppose further that the sutures $\widehat{\Gamma}_n$ and $\widehat{\Gamma}_\mu$ are defined as in Definition 5.1.2. Then there are two exact triangles related to $\psi^*_{+,*}$ and $\psi^*_{-,*}$,* *respectively.*

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(-Y\backslash K,-\widehat{\Gamma}_n) & \xrightarrow{\psi^{n}_{\pm,n+1}} & \underline{\mathrm{SHI}}(-Y\backslash K,-\widehat{\Gamma}_{n+1}) \\ {\scriptstyle \psi^{\mu}_{\pm,n}}\nwarrow & & \swarrow{\scriptstyle \psi^{n+1}_{\pm,\mu}} \\ & \underline{\mathrm{SHI}}(-Y\backslash K,-\widehat{\Gamma}_\mu) & \end{array} \tag{5.1.1}$$

Proof. If $\widehat{\Gamma}_{n+1}=\gamma_{(x_3,y_3)}$ and $y_3>-x_3>0$, then it is straightforward to check that

$$\gamma_{(x_1,y_1)}=\widehat{\Gamma}_\mu \text{ and } \gamma_{(x_2,y_2)}=\widehat{\Gamma}_n,$$

where (x_1,y_1) and (x_2,y_2) are defined as in Definition 5.1.4. Then the exact triangles follows from Proposition 5.1.6. The similar proof applies to other cases. □

The bypass maps in (5.1.1) behave well under the gradings on $\underline{\mathrm{SHI}}$ associated to the fixed Seifert surface of K. To provide more details, let us fix a minimal genus Seifert surface S of K that always has minimal possible intersections with any suture $\gamma_{(p,q)}$. Hence $g(S)=g(K)$. We consider the $\mathbb{Z}$-grading (or $(\mathbb{Z}+\frac{1}{2})$-grading) associated to S (*c.f.* Subsection 2.3.3).

Lemma 5.1.8. *Suppose $K\subset Y$ is a null-homologous knot and $\gamma_{(x,y)}$ is a suture on $\partial Y\backslash K$ with $y\geq 0$. Suppose further that S is a minimal genus Seifert surface of K. Then the maximal and minimal nontrivial gradings of $\underline{\mathrm{SHI}}(-Y\backslash K,-\gamma_{(x,y)},S)$ are*

$$i_{max}=\frac{y-1}{2}+g(K)$$

and

$$i_{min}=-\frac{y-1}{2}-g(K).$$

Proof. The notations i_{max} and i_{min} are used in Subsection 3.1.2, while now we could identify the top and bottom nontrivial gradings by making use of sutured manifold decompositions in Term (2) of Theorem 2.3.20. Note that we have assumed that the knot complement $Y\backslash K$ is irreducible in the convention after Definition 5.1.1, and S is a minimal genus Seifert surface of K, so the decomposition of $(Y\backslash K,\gamma)$ along S and $-S$ are both taut. □

Definition 5.1.9. For any integer $y\in\mathbb{N}$, define

$$i^y_{max}=\frac{y-1}{2}+g(K) \text{ and } i^y_{min}=-\frac{y-1}{2}-g(K).$$

For the suture $\widehat{\Gamma}_n = \gamma_{(p_n,q_n)}$, define

$$\hat{i}^n_{max} = i^{q_n}_{max} \text{ and } \hat{i}^n_{min} = i^{q_n}_{min}.$$

The following lemma is similar to Lemma 3.1.6.

Proposition 5.1.10. *Suppose $K \subset Y$ is a null-homologous knot and suppose the pair $(\widehat{Y}, \widehat{K})$ is obtained from (Y,K) by a q/p surgery. Suppose further that the sutures $\widehat{\Gamma}_n$ and $\widehat{\Gamma}_\mu$ are defined as in Definition 5.1.2 and S is a minimal genus Seifert surface of K. Then the following hold. Note that the grading shift notation comes from Definition 3.1.5.*

(1) For $n \in \mathbb{Z}$ so that $q_{n+1} = q_n + q$, i.e., $n \geq 0$, there are two bypass exact triangles:

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_n, S)[\hat{i}^{n+1}_{min} - \hat{i}^n_{min}] & \xrightarrow{\psi^n_{+,n+1}} & \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{n+1}, S) \\ \uparrow {\scriptstyle \psi^{\mu}_{+,n}} & \swarrow {\scriptstyle \psi^{n+1}_{+,\mu}} & \\ \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_\mu, S)[\hat{i}^{n+1}_{max} - \hat{i}^{\mu}_{max}] & & \end{array}$$

and

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_n, S)[\hat{i}^{n+1}_{max} - \hat{i}^n_{max}] & \xrightarrow{\psi^n_{-,n+1}} & \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{n+1}, S) \\ \uparrow {\scriptstyle \psi^{\mu}_{-,n}} & \swarrow {\scriptstyle \psi^{n+1}_{-,\mu}} & \\ \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_\mu, S)[\hat{i}^{n+1}_{min} - \hat{i}^{\mu}_{min}] & & \end{array}$$

(2) For $n \in \mathbb{Z}$ so that $q_{n+1} = q_n - q$, i.e., $n < -1$, there are two bypass exact triangles:

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_n, S) & \xrightarrow{\psi^n_{+,n+1}} & \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{n+1}, S)[\hat{i}^{n}_{max} - \hat{i}^{n+1}_{max}] \\ \uparrow {\scriptstyle \psi^{\mu}_{+,n}} & \swarrow {\scriptstyle \psi^{n+1}_{+,\mu}} & \\ \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_\mu, S)[\hat{i}^{n}_{min} - \hat{i}^{\mu}_{min}] & & \end{array}$$

and

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_n, S) & \xrightarrow{\psi^n_{-,n+1}} & \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{n+1}, S)[\hat{i}^{n}_{min} - \hat{i}^{n+1}_{min}] \\ \uparrow {\scriptstyle \psi^{\mu}_{-,n}} & \swarrow {\scriptstyle \psi^{n+1}_{-,\mu}} & \\ \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_\mu, S)[\hat{i}^{n}_{max} - \hat{i}^{\mu}_{max}] & & \end{array} \tag{5.1.2}$$

(3) For $n \in \mathbb{Z}$ so that $q_{n+1} + q_n = q$, i.e., $n = -1$, there are two bypass exact triangles:

$$
\begin{array}{ccc}
\underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_n, S)\,[\hat{i}^{\mu}_{min} - \hat{i}^{n}_{min}] & \xrightarrow{\psi^{n}_{+,n+1}} & \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{n+1}, S)\,[\hat{i}^{\mu}_{max} - \hat{i}^{n+1}_{max}] \\
\uparrow{\scriptstyle \psi^{\mu}_{+,n}} & \swarrow{\scriptstyle \psi^{n+1}_{+,\mu}} & \\
\underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{\mu}, S) & &
\end{array}
\tag{5.1.3}
$$

and

$$
\begin{array}{ccc}
\underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_n, S)\,[\hat{i}^{\mu}_{max} - \hat{i}^{n}_{max}] & \xrightarrow{\psi^{n}_{-,n+1}} & \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{n+1}, S)\,[\hat{i}^{\mu}_{min} - \hat{i}^{n+1}_{min}] \\
\uparrow{\scriptstyle \psi^{\mu}_{-,n}} & \swarrow{\scriptstyle \psi^{n+1}_{-,\mu}} & \\
\underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{\mu}, S) & &
\end{array}
\tag{5.1.4}
$$

Furthermore, all maps involved in the above bypass exact triangles preserve the gradings induced by surfaces.

Remark 5.1.11. We can understand the above proposition by the following diagramatic method, which is inspired by the curve invariant introduced by Hanselman, Rasmussen, and Waston [HRW17, HRW18].

(1) Consider the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$. A surgery slope $y/x \in \mathbb{Q} \cup \{\infty\}$ corresponds to a straight arc connecting two lattice points in $\mathbb{Z}^2$.

(2) Suppose the sutures $\gamma_{(x_1,y_1)}$, $\gamma_{(x_2,y_2)}$, and $\gamma_{(x_3,y_3)}$ are defined as in Definition 5.1.4. Then it is easy to see the arcs corresponding to these three sutures bound a triangle containing no lattice point in the interior. There are two different triangles up to translation, which correspond to two different bypass triangles. All bypass maps are clockwise in $\mathbb{R}^2$. Rotation around the origin by 180 degrees will switch the roles of $\psi^{*}_{+,*}$ and $\psi^{*}_{-,*}$.

(3) The height of the middle point of the straight arc indicates the grading before stabilization (so there are gradings of half integers). If the top endpoints of two arcs are the same, the grading shift is about $\hat{i}^{*}_{min}$. If the bottom endpoints of two arcs are the same, the grading shift is about $\hat{i}^{*}_{max}$.

The following lemmas are special cases of results in previous Chapters for knot complements. We may abuse the notations for bypass maps so they also denote the restrictions on some gradings associated to S.

Lemma 5.1.12 (Lemma 3.1.7)**.** *For any $n \in \mathbb{N}$, the map*

$$\psi^n_{+,n+1} : \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_n, S, i) \to \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{n+1}, S, i - \hat{i}^n_{min} + \hat{i}^{n+1}_{min})$$

is an isomorphism if $i \le \hat{i}^n_{max} - 2g(K)$. Similarly, the map

$$\psi^n_{-,n+1} : \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_n, S, i) \to \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{n+1}, S, i - \hat{i}^n_{max} + \hat{i}^{n+1}_{max})$$

is an isomorphism if $i \ge \hat{i}^n_{min} + 2g(K)$.

Lemma 5.1.13 (Lemma 4.2.1)**.** *Suppose $n \in \mathbb{N}$ satisfies $q_n \ge q + 2g(K)$, and suppose $i, j \in \mathbb{Z}$ with*

$$\hat{i}^n_{min} + 2g(K) \le i, j \le \hat{i}^n_{max} - 2g(K), \text{ and } i - j = q.$$

Then we have

$$\underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_n, S, i) \cong \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_n, S, j).$$

Thus, we can divide $\underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_n)$ into three parts: the top $2g(K)$ gradings, the middle gradings, and the bottom $2g(K)$ gradings. All parts stabilize by Lemma 5.1.12 and the spaces in the middle gradings are cyclic by Lemma 5.1.13. Moreover, by Theorem 2.3.20, we have a canonical isomorphism

$$\underline{\mathrm{SHI}}(-M, -\gamma, S, i) \cong \underline{\mathrm{SHI}}(-M, \gamma, S, -i).$$

If $\partial M \cong T^2$, we can identify $-\gamma$ with γ, which induces an isomorphism

$$\iota_\gamma : \underline{\mathrm{SHI}}(-M, -\gamma, S, i) \xrightarrow{\cong} \underline{\mathrm{SHI}}(-M, \gamma, S, -i) \xrightarrow{=} \underline{\mathrm{SHI}}(-M, -\gamma, S, -i). \tag{5.1.5}$$

Hence the spaces in the top $2g(K)$ gradings and the bottom $2g(K)$ gradings are isomorphic.

The following theorems imply that spaces in the middle gradings encode information of $I^\sharp(-\widehat{Y})$.

Lemma 5.1.14 (Lemma 3.1.8)**.** *Suppose $K \subset Y$ is a null-homologous knot and suppose the pair $(\widehat{Y}, \widehat{K})$ is obtained from (Y, K) by a q/p surgery. Suppose further that the sutures $\widehat{\Gamma}_n$ are defined as in Definition 5.1.2. Then, there is an exact triangle*

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_n) & \longrightarrow & \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{n+1}) \\ {\scriptstyle G_n}\nwarrow & & \swarrow{\scriptstyle F_{n+1}} \\ & I^\sharp(-\widehat{Y}) & \end{array} \tag{5.1.6}$$

where F_n is the contact gluing maps associated to the contact 2-handle attachment along $\hat{\mu} = q\mu + p\lambda \subset \partial Y\backslash K$. Furthermore, we have four commutative diagrams related to $\psi^n_{+,n+1}$ and $\psi^n_{-,n+1}$, respectively

$$\begin{array}{ccc}
\underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_n) & \xrightarrow{\psi^n_{\pm,n+1}} & \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{n+1}) \\
{\scriptstyle G_n}\nwarrow & & \nearrow{\scriptstyle G_{n+1}} \\
& I^\sharp(-\widehat{Y}) &
\end{array}$$

and

$$\begin{array}{ccc}
\underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_n) & \xrightarrow{\psi^n_{\pm,n+1}} & \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{n+1}) \\
{\scriptstyle F_n}\searrow & & \swarrow{\scriptstyle F_{n+1}} \\
& I^\sharp(-\widehat{Y}) &
\end{array}$$

Theorem 5.1.15 (Proposition 4.2.10)**.** *Suppose $n \in \mathbb{N}$ satisfies $q_n \geq q + 2g(K)$. Then there exists an isomorphism*

$$F_n' : \bigoplus_{i=0}^{q-1} \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_n, S, \hat{i}^n_{max} - 2g(K) - i) \xrightarrow{\cong} I^\sharp(-\widehat{Y}),$$

where F_n' is the restriction of F_n in Lemma 5.1.14.

Definition 5.1.16 (Definition 4.2.2)**.** For a fixed integer $q > 0$ and any integer $s \in [0, q-1]$, suppose $[s]$ is the image of s in $\mathbb{Z}_q$. Define

$$I^\sharp(-\widehat{Y}, [s]) := F_n'(\underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_n, S, \hat{i}^n_{max} - 2g(K) - s)) \subset I^\sharp(-\widehat{Y}).$$

It is well-defined by isomorphisms in Lemma 5.1.12 and commutative diagrams in Lemma 5.1.14.

Proposition 5.1.17. *Suppose K is a knot in an integral homology sphere Y and suppose n is an integer. Then $-Y_{-n}(K)$ is an instanton L-space if and only if for any $[s] \in \mathbb{Z}_{|n|}$, we have*

$$\dim_{\mathbb{C}} I^\sharp(-Y_{-n}(K), [s]) = 1.$$

Proof. It follows from the special case $(M, \gamma) = (Y(1), \delta)$ in Theorem 1.2.1:

$$\chi_{\mathrm{en}}(I^\sharp(Y)) = \chi(\widehat{HF}(Y)) = \sum_{h \in H_1(Y)} h \in \mathbb{Z}[H_1(Y)]/\pm H_1(Y),$$

where Y is any rational homology sphere. □

5.1.3 Commutative diagrams for bypass maps

In this subsection, we show there are some commutative diagrams for bypass maps.

Lemma 5.1.18 ([Li19, Corollary 2.20])**.** *For any surgery slope q/p, consider the bypass maps $\psi^*_{+,*}$ and $\psi^*_{-,*}$ in Proposition 5.1.7. For any integer $n \in \mathbb{Z}$, we have the following commutative diagram.*

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(-Y(K),-\widehat{\Gamma}_n) & \xrightarrow{\psi^n_{-,n+1}} & \underline{\mathrm{SHI}}(-Y(K),-\widehat{\Gamma}_{n+1}) \\ \Big\downarrow{\scriptstyle \psi^n_{+,n+1}} & & \Big\downarrow{\scriptstyle \psi^{n+1}_{+,n+2}} \\ \underline{\mathrm{SHI}}(-Y(K),-\widehat{\Gamma}_{n+1}) & \xrightarrow{\psi^{n+1}_{-,n+2}} & \underline{\mathrm{SHI}}(-Y(K),-\widehat{\Gamma}_{n+2}) \end{array} \tag{5.1.7}$$

Proof. In Subsection 2.3.4, we interpreted bypass maps by contact gluing maps. So the composition of bypass maps becomes the composition of contact gluing maps. To verify the commutative diagram, it suffices to verify that two contact structures coming from different bypasses are actually the same. Thus, it is free to change the basis of $H_1(T^2)$. It suffices to verify a special case $q/p = 1/0$ and $n = 0$. Then it follows from [Hon00, Lemma 4.14] that the contact structures are the same.

□

Lemma 5.1.19. *For any surgery slope q/p, consider the bypass maps $\psi^*_{+,*}$ and $\psi^*_{-,*}$ in Proposition 5.1.7. For any $n \in \mathbb{Z}$, we have two commutative diagrams*

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(-Y(K),-\widehat{\Gamma}_n) & \xrightarrow{\psi^n_{-,n+1}} & \underline{\mathrm{SHI}}(-Y(K),-\widehat{\Gamma}_{n+1}) \\ & {\scriptstyle \psi^n_{+,\mu}}\searrow \quad \swarrow{\scriptstyle \psi^{n+1}_{+,\mu}} & \\ & \underline{\mathrm{SHI}}(-\widehat{Y}(K),-\widehat{\Gamma}_\mu) & \end{array} \tag{5.1.8}$$

and

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(-Y(K),-\widehat{\Gamma}_n) & \xrightarrow{\psi^n_{-,n+1}} & \underline{\mathrm{SHI}}(-Y(K),-\widehat{\Gamma}_{n+1}) \\ & {\scriptstyle \psi^\mu_{+,n}}\nwarrow \quad \nearrow{\scriptstyle \psi^\mu_{+,n+1}} & \\ & \underline{\mathrm{SHI}}(-\widehat{Y}(K),-\widehat{\Gamma}_\mu) & \end{array} \tag{5.1.9}$$

The similar commutative diagrams hold if we switch the roles of $\psi^*_{+,*}$ *and* $\psi^*_{-,*}$.

Remark 5.1.20. The bypass maps in Lemma 5.1.19 are from different bypass exact triangles. For example, the map $\psi^{\mu}_{+,n}$ is in the triangle involving $\widehat{\Gamma}_\mu, \widehat{\Gamma}_n$, and $\widehat{\Gamma}_{n+1}$ while the map $\psi^{n}_{+,\mu}$ is in the triangle involving $\widehat{\Gamma}_\mu, \widehat{\Gamma}_{n-1}$, and $\widehat{\Gamma}_n$, where the superscripts in the notations of bypass maps denote the sources the subscripts denote the targets.

Proof of Lemma 5.1.19. Similar to the proof of Lemma 5.1.18, this lemma follows from Honda's classification of tight contact structures on $T^2 \times I$ [Hon00, Lemma 4.14].

□

Corollary 5.1.21. *For any surgery slope* q/p*, consider the bypass maps* $\psi^*_{+,*}$ *and* $\psi^*_{-,*}$ *in Proposition 5.1.7. For any* $i, j \in \mathbb{Z}$*, we have the following commutative diagrams related to* $\psi^*_{+,*}$ *and* $\psi^*_{-,*}$*, respectively.*

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(-Y(K),-\widehat{\Gamma}_\mu) & \xrightarrow{\psi^{\mu}_{\pm,j}} & \underline{\mathrm{SHI}}(-Y(K),-\widehat{\Gamma}_j) \\ \downarrow \psi^{\mu}_{\pm,i} & & \downarrow \psi^{j}_{\pm,\mu} \\ \underline{\mathrm{SHI}}(-Y(K),-\widehat{\Gamma}_i) & \xrightarrow{\psi^{i}_{\pm,\mu}} & \underline{\mathrm{SHI}}(-Y(K),-\widehat{\Gamma}_\mu) \end{array} \tag{5.1.10}$$

Proof. The commutative diagram related to $\psi^*_{+,*}$ follows from (5.1.8) and (5.1.9). Explicitly, for $i = j+1$, both compositions of maps are equal to

$$\psi^{j+1}_{+,\mu} \circ \psi^{n}_{-,j+1} \circ \psi^{\mu}_{+,j}.$$

The other commutative diagram follows from Lemma 5.1.19 similarly. □

Corollary 5.1.22. *For any surgery slope* q/p*, consider the bypass maps* $\psi^*_{+,*}$ *and* $\psi^*_{-,*}$ *in Proposition 5.1.7. For any* $n \in \mathbb{Z}$*, we have*

$$\psi^{n}_{+,\mu} \circ \psi^{\mu}_{-,n} = \psi^{n}_{-,\mu} \circ \psi^{\mu}_{+,n} = 0$$

and

$$\psi^{\mu}_{+,n} \circ \psi^{n}_{+,\mu} = \psi^{\mu}_{-,n} \circ \psi^{n}_{-,\mu} = 0$$

Proof. By Lemma 5.1.19 and the exactness, we have

$$\psi^{n}_{+,\mu} \circ \psi^{\mu}_{-,n} = \psi^{n+1}_{+,\mu} \circ \psi^{n}_{-,n+1} \circ \psi^{\mu}_{-,n} = 0.$$

Other arguments follow from Lemma 5.1.19 and the exactness similarly. □

Remark 5.1.23. The above commutative diagrams can be illustrated by the method described in Remark 5.1.11. The illustration of the special cases in the proofs is shown in Figure 5.2. Note that vector spaces are denoted by their sutures (we omit the minus signs), and all maps are bypass maps. They are grading preserving and commute with F_* and G_* by Proposition 5.1.10 and Lemma 5.1.14, respectively.

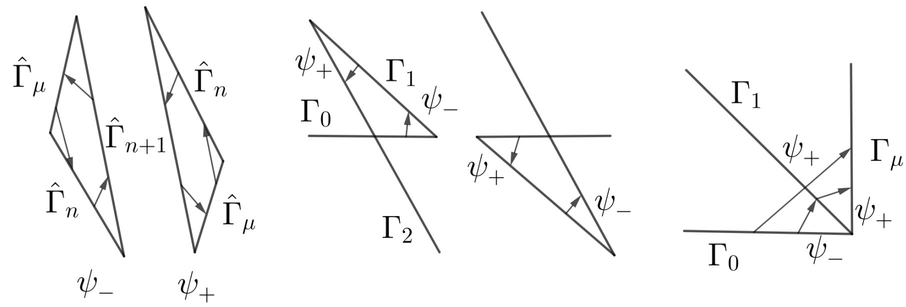


Figure 5.2 Left, bypass maps; middle, illustration of (5.1.7); right, illustration of (5.1.8).

5.1.4 Two spectral sequences

In this subsection, we construct spectral sequences from $\underline{\mathrm{KHI}}(-\widehat{Y},\widehat{K})$ to $I^\sharp(-\widehat{Y})$ by bypass exact triangles in Proposition 5.1.10.

For a fixed integer $q > 0$, any fixed large integer n, and any integer i, we have the following diagram of exact triangles

$$\begin{array}{ccccccccccccc}
\cdots \leftarrow & \widehat{\Gamma}^{i,+}_{n+1} & & \xleftarrow{\psi^{n}_{+,n+1}} & & \widehat{\Gamma}^{i,+}_{n} & & \xleftarrow{\psi^{n-1}_{+,n}} & & \widehat{\Gamma}^{i,+}_{n-1} & \xleftarrow{\psi^{n-2}_{+,n-1}} & \widehat{\Gamma}^{i,+}_{n-2} & \leftarrow \cdots \\
& & \searrow^{\psi^{n+1}_{+,\mu}} & & \nearrow^{\psi^{\mu}_{+,n}} & & \searrow^{\psi^{n}_{+,\mu}} & & \nearrow^{\psi^{\mu}_{+,n-1}} & & \searrow^{\psi^{n-1}_{+,\mu}} \quad \nearrow^{\psi^{\mu}_{+,n-2}} & & \\
& \cdots & & \widehat{\Gamma}^{i-q}_{\mu} & & & & \widehat{\Gamma}^{i}_{\mu} & & & \widehat{\Gamma}^{i+q}_{\mu} & \cdots & \\
& & \swarrow_{\psi^{\mu}_{-,n-2}} & & \nwarrow_{\psi^{n-1}_{-,\mu}} & & \swarrow_{\psi^{\mu}_{-,n-1}} & & \nwarrow_{\psi^{n}_{-,\mu}} & & \swarrow_{\psi^{\mu}_{-,n}} \quad \nwarrow_{\psi^{n+1}_{-,\mu}} & & \\
\cdots \rightarrow & \widehat{\Gamma}^{i,-}_{n-2} & & \xrightarrow[\psi^{n-2}_{-,n-1}]{} & & \widehat{\Gamma}^{i,-}_{n-1} & & \xrightarrow[\psi^{n-1}_{-,n}]{} & & \widehat{\Gamma}^{i,-}_{n} & \xrightarrow[\psi^{n}_{-,n+1}]{} & \widehat{\Gamma}^{i,-}_{n+1} & \rightarrow \cdots
\end{array} \tag{5.1.11}$$

where we write

$$\begin{aligned}
\widehat{\Gamma}^{i}_{\mu} &= \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_\mu, S, i) \\
\widehat{\Gamma}^{i,+}_{k} &= \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_k, S, i + \hat{i}^k_{min} - \hat{i}^n_{min} + \hat{i}^n_{max} - \hat{i}^\mu_{max}) \\
\widehat{\Gamma}^{i,-}_{k} &= \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_k, S, i + \hat{i}^k_{max} - \hat{i}^n_{max} + \hat{i}^n_{min} - \hat{i}^\mu_{min})
\end{aligned}$$

for any $k \in \mathbb{N}$, and we abuse notations so that the maps $\psi^*_{+,*}, \psi^*_{-,*}$ also denote the restrictions on corresponding gradings. Note that $\hat{i}^*_{max}$ and $\hat{i}^*_{min}$ are the maximal and minimal nontrivial gradings of $\underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_*)$ associated to S, respectively. By direct calculation, we have

$$\widehat{\Gamma}^{i,+}_{n+k} \cong \widehat{\Gamma}^{i,+}_{n+k-1} \text{ for } k > \frac{i-\hat{i}^\mu_{min}}{q} \text{ and } \widehat{\Gamma}^{i,+}_{n-k} = 0 \text{ for } -k < \frac{i-\hat{i}^\mu_{max}}{q}, \tag{5.1.12}$$

$$\widehat{\Gamma}^{i,-}_{n+k} \cong \widehat{\Gamma}^{i,-}_{n+k-1} \text{ for } k > \frac{\hat{i}^\mu_{max}-i}{q} \text{ and } \widehat{\Gamma}^{i,-}_{n-k} = 0 \text{ for } -k < \frac{\hat{i}^\mu_{min}-i}{q}. \tag{5.1.13}$$

Theorem 5.1.24. *There exist two spectral sequences $\{(E_{r,+}, d_{r,+})\}_{r\geq 1}$ and $\{(E_{r,-}, d_{r,-})\}_{r\geq 1}$ with*

$$E_{1,+} = E_{1,-} = \underline{\mathrm{KHI}}(-\widehat{Y}, \widehat{K})$$

induced by exact triangles in (5.1.11) involving $\psi^*_{+,*}$ and $\psi^*_{-,*}$, respectively. They are independent of the choice of the integer n. Suppose $\{(E_{r,\pm}, d_{r,\pm})\}_{r\geq 1}$ converge to $\mathcal{G}_\pm$, respectively. Then there are isomorphisms

$$\mathcal{G}_\pm \cong I^\sharp(-\widehat{Y}).$$

Proof. The proof is based on unrolled exact couples introduced in Subsection 2.2.2.

The exact triangles about $\psi^*_{+,*}$ form an unrolled exact couple in the sense of Definition 2.2.3. For simplicity, we consider the direct sum of the unrolled exact couples about $i = i_0+1, \ldots, i_0+q$ for some i_0 so that $i \in [\hat{i}^\mu_{min}, \hat{i}^\mu_{max}]$. Then the first page is the same as

$$\underline{\mathrm{KHI}}(-\widehat{Y}, \widehat{K}) = \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_\mu)$$

Since there are only finitely many nontrivial gradings of associated to S, this unrolled exact couple is bounded. Proposition 2.2.5 provides a spectral sequence $\{(E_{r,+}, d_{r,+})\}_{r\geq 1}$ with $E_{1,+} = \underline{\mathrm{KHI}}(-\widehat{Y}, \widehat{K})$.

Since

$$\hat{i}^k_{max} - \hat{i}^k_{min} = kq - q_0 - 1 + 2g(K) \text{ and } \hat{i}^\mu_{max} - \hat{i}^\mu_{min} = q - 1 + 2g(K),$$

for any integers $i \geq \hat{i}^{\mu}_{min}$ and $k < n - (q-1+2g(K))/q$, we have

$$\begin{aligned}(i+\hat{i}^{k}_{min}-\hat{i}^{n}_{min}+\hat{i}^{n}_{max}-\hat{i}^{\mu}_{max})-\hat{i}^{k}_{max} &= i+(\hat{i}^{k}_{min}-\hat{i}^{k}_{max})+(\hat{i}^{n}_{max}-\hat{i}^{n}_{min})-\hat{i}^{\mu}_{max}\\ &= i-(kq-q_0-1+2g(K))+(nq-q_0-1+2g(K))-\hat{i}^{\mu}_{max}\\ &= i+(n-k)q-\hat{i}^{\mu}_{max}\\ &\geq \hat{i}^{\mu}_{min}+(n-k)q-\hat{i}^{\mu}_{max}\\ &= (n-k)q-(q-1+2g(K))\\ &> 0.\end{aligned} \tag{5.1.14}$$

For such k, we have $\widehat{\Gamma}^{i,+}_{k}=0$. Thus, by Theorem 2.2.6, we know that $\{(E_{r,+},d_{r,+})\}_{r\geq 1}$ converges to

$$\mathcal{G}_{+}=\bigoplus_{i=i_0+1}^{i_0+q}\widehat{\Gamma}^{i,+}_{n+l}\subset \underline{\mathrm{SHI}}(-Y\backslash K,-\widehat{\Gamma}_{n+l})$$

for some large integer l. The calculation in (5.1.14) also indicates that $\mathcal{G}_{+}$ lives in the middle gradings of $\underline{\mathrm{SHI}}(-Y\backslash K,-\widehat{\Gamma}_{n+l})$. Hence by Lemma 5.1.13 and Theorem 5.1.15, we know that $\mathcal{G}_{+}\cong I^{\sharp}(-\widehat{Y})$. The independence of the integer n follows from Lemma 5.1.12 and Lemma 5.1.19. The maps $\psi^{*}_{-,*}$ induces an isomorphism between spectral sequences since they induce an isomorphism between the first pages.

Similar argument applies to exact triangles involving $\psi^{*}_{-,*}$ and we obtain another spectral sequence $\{(E_{r,-},d_{r,-})\}_{r\geq 1}$ with $E_{1,-}=\underline{\mathrm{KHI}}(-\widehat{Y},\widehat{K})$, which converges to

$$\mathcal{G}_{-}\subset \underline{\mathrm{SHI}}(-Y\backslash K,-\widehat{\Gamma}_{n+l})$$

in middle gradings for some large integer l. Also, we have $\mathcal{G}_{-}\cong I^{\sharp}(-\widehat{Y})$. □

5.1.5 Bent complexes

In this subsection, we construct the bent complex and relate its homology to negative large surgeries. The construction and the name are inspired by Heegaard Floer theory (*c.f.* [Ras07, Section 4.1], [RR17, Section 2.2]; see also [OS04b, Section 4]).

Construction 5.1.25. Suppose $\hat{\mu}=q\mu+p\lambda$. Consider the spectral sequences $\{(E_{r,+},d_{r,+})\}_{r\geq 1}$ and $\{(E_{r,-},d_{r,-})\}_{r\geq 1}$ constructed in Theorem 5.1.24. By fixing a basis of $\underline{\mathrm{KHI}}(-\widehat{Y},\widehat{K})$, Construction 2.2.7 provides two filtered chain complexes

$$(\underline{\mathrm{KHI}}(-\widehat{Y},\widehat{K}),d_{+})\text{ and }(\underline{\mathrm{KHI}}(-\widehat{Y},\widehat{K}),d_{-})$$

such that the induced spectral squences are $\{(E_{r,+},d_{r,+})\}_{r\ge1}$ and $\{(E_{r,-},d_{r,-})\}_{r\ge1}$, respectively. For any integer s, the **bent complex** is

$$A_s=A_s(-Y,K):=(\bigoplus_{k\in\mathbb{Z}}\underline{\mathrm{SHI}}(-Y\backslash K,-\widehat{\Gamma}_\mu,S,s+kq),d_s),$$

where for any element $x\in\underline{\mathrm{SHI}}(-Y\backslash K,-\widehat{\Gamma}_\mu,S,s+kq)$,

$$d_s(x)=\begin{cases}d_+(x) & k>0,\\ d_+(x)+d_-(x) & k=0,\\ d_-(x) & k<0.\end{cases}$$

It is easy to check $d_s\circ d_s=0$.

Remark 5.1.26. Since $\underline{\mathrm{SHI}}$ is a projectively transitive system, the maps $d_{r,+}$ and $d_{r,-}$ only well-defined up to multiplication of a unit. However, the kernel and the image of a map are still well-defined, so we can still define exact sequences for projectively transitive systems. Moreover, if $f:A\to B$ and $g:A\to C$ are maps between projectively transitive systems, though the map

$$f+g:=f\oplus g=(f,g):A\to B\oplus C$$

is not well-defined, its kernel ($\mathrm{Ker}\,f\cap\mathrm{Ker}\,g$) is well-defined, so there is no ambiguity to consider the dimension of the homology of the bent complex. Alternatively, by Remark 2.2.2 and the discussion after Theorem 2.3.16, we can always fix closures of corresponding balanced sutured manifolds and consider linear maps between actual vector spaces, at the cost that equations between maps only hold up to multiplication by a unit.

The main theorem of this subsection is the following.

Theorem 5.1.27. *Suppose $\hat{\mu}=q\mu+p\lambda$ with $q\in\mathbb{N}_+$. For any integer s, let $H(A_s)$ denote the homology of the bent complex A_s in Construction 5.1.25. For any integer n satisfying $(n-1)q\ge 2g(K)$, we have an isomorphism for some integer j_n:*

$$a_{s,n}:H(A_s)\xrightarrow{\cong}\underline{\mathrm{SHI}}(-Y\backslash K,-\gamma_{2\hat{\lambda}-(2n-1)\hat{\mu}},S,s+j_n). \tag{5.1.15}$$

Suppose the maximal and minimal nontrivial gradings of $\underline{\mathrm{SHI}}(-Y\backslash K,-\gamma_{2\hat{\lambda}-(2n-1)\hat{\mu}})$ are $\hat{i}^{\sharp}_{max}$ and $\hat{i}^{\sharp}_{min}$, which can be calculated by Lemma 5.1.8. Then we have

$$j_n=\hat{i}^{\sharp}_{min}-\hat{i}^{n}_{min}+\hat{i}^{n}_{max}-\hat{i}^{\mu}_{max}=\hat{i}^{\sharp}_{max}-\hat{i}^{n}_{max}+\hat{i}^{n}_{min}-\hat{i}^{\mu}_{min}.$$

Remark 5.1.28. By Definition 5.1.9, we have $i^y_{max} - i^y_{min} = 2g(K) + y - 1$. Then

$$\begin{aligned}(\hat{i}^{\sharp}_{min} - &\hat{i}^{n}_{min} + \hat{i}^{n}_{max} - \hat{i}^{\mu}_{max}) - (\hat{i}^{\sharp}_{max} - \hat{i}^{n}_{max} + \hat{i}^{n}_{min} - \hat{i}^{\mu}_{min}) \\ &= 2(\hat{i}^{n}_{max} + \hat{i}^{n}_{min} -) - (\hat{i}^{\sharp}_{max} - \hat{i}^{\sharp}_{min}) - (\hat{i}^{\mu}_{max} - \hat{i}^{\mu}_{min}) \\ &= 2(nq - q_0 - 1) - ((2n-1)q - 2q_0 - 1) - (q-1) \\ &= 0.\end{aligned}$$

Hence j_n in Theorem 5.1.27 is well-defined.

Proof of Theorem 5.1.27. We consider two cases. The first case is special, and we use the octahedral axiom to prove it. The second case is more general, and we reduce it to the first case. For the bent complex A_s, we fix $i = s$ in the diagram (5.1.11).

Case 1. Suppose $\widehat{\Gamma}^{i,+}_k = \widehat{\Gamma}^{i,-}_k = 0$ for $k \le n-2$ in the diagram (5.1.11).

In this case, higher differentials $d_{r,\pm}$ for $r \ge 2$ vanish and the maps

$$\psi^{n-1}_{\pm,\mu} : \widehat{\Gamma}^{i,\pm}_{n-1} \to \widehat{\Gamma}^{i\pm q}_{\mu}$$

are isomorphisms. Hence

$$A_s = (\widehat{\Gamma}^{i}_{\mu} \oplus \widehat{\Gamma}^{i,+}_{n-1} \oplus \widehat{\Gamma}^{i,-}_{n-1}, f),$$

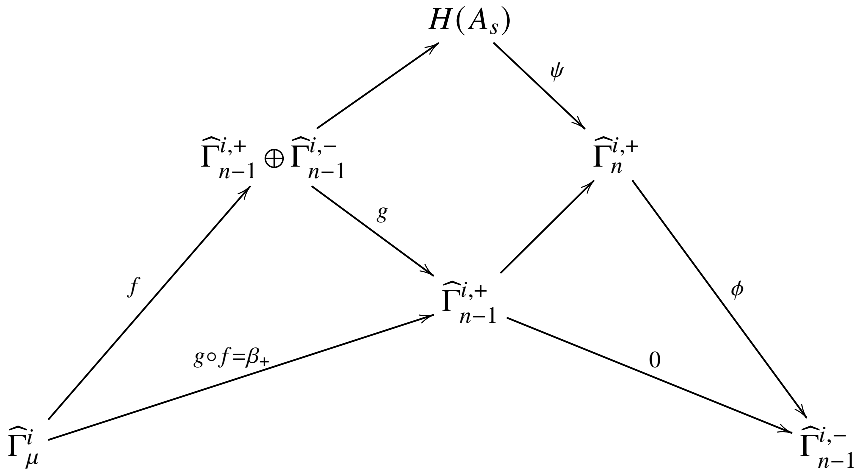
where

$$\begin{aligned} f : \widehat{\Gamma}^{i}_{\mu} &\to \widehat{\Gamma}^{i,+}_{n-1} \oplus \widehat{\Gamma}^{i,-}_{n-1} \\ f(x) &= (\beta_+(x), \beta_-(x)) \end{aligned}$$

is the restriction of $(\psi^{\mu}_{+,n-1}(x), \psi^{\mu}_{-,n-1}(x))$. Define $g : \widehat{\Gamma}^{i,+}_{n-1} \oplus \widehat{\Gamma}^{i,-}_{n-1} \to \widehat{\Gamma}^{i,+}_{n-1}$ to be the projection map. Then we apply Lemma 2.2.9 to

$$X = \widehat{\Gamma}^{i}_{\mu}, Y = \widehat{\Gamma}^{i,+}_{n-1} \oplus \widehat{\Gamma}^{i,-}_{n-1}, Z = \widehat{\Gamma}^{i,+}_{n-1}, X' = \widehat{\Gamma}^{i,-}_{n-1}, Y' = \widehat{\Gamma}^{i,+}_{n}, Z' = H(A_s).$$

Then there exist maps ψ and ϕ making the following diagram commute and exact



Thus, we obtain a long exact sequence

$$\cdots \longrightarrow H(A_s) \xrightarrow{\psi} \widehat{\Gamma}^{i,+}_{n} \xrightarrow{\phi} \widehat{\Gamma}^{i,-}_{n-1} \longrightarrow H(A_s)\{1\} \longrightarrow \cdots$$

Let

$$\alpha_+ : \widehat{\Gamma}^{i,+}_{n} \longrightarrow \widehat{\Gamma}^{i}_{\mu}$$

be the restriction of $\psi^{n}_{+,\mu}$. Note that

$$\widehat{\Gamma}^{i,+}_{n} \cong \mathrm{Im}(\psi^{n-1}_{+,n} : \widehat{\Gamma}^{i,+}_{n-1} \to \widehat{\Gamma}^{i,+}_{n}) \oplus \mathrm{Coker}(\psi^{n-1}_{+,n} : \widehat{\Gamma}^{i,+}_{n-1} \to \widehat{\Gamma}^{i,+}_{n}) \cong \mathrm{Ker}(\beta_+) \oplus \mathrm{Coker}(\beta_+).$$

By results in Subsection 5.1.3, We know the maps ϕ and $\phi' := \beta_- \circ \alpha_+$ satisfying the assumption of Lemma 2.2.10. Thus, we have

$$H(A_s) \cong H(\mathrm{Cone}(\phi)) \cong H(\mathrm{Cone}(\beta_- \circ \alpha_+)). \tag{5.1.16}$$

Note that we assume $\hat{\mu} = q\mu + p\lambda$ for $q \geq 0$ and $\hat{\lambda} = q_0\mu + p_0\lambda$ satisfying Definition 5.1.2. When n is large, the coefficient of μ in

$$\hat{\mu}' := n\hat{\mu} - \hat{\lambda} = (nq - q_0)\mu + (np - p_0)\lambda$$

is positive. By Definition 5.1.2 we set

$$\hat{\lambda}' := \hat{\lambda} - (n-1)\hat{\mu} = (q_0 - (n-1)q)\mu + (p_0 - (n-1)p)\lambda.$$

Then

$$\hat{\lambda}' + \hat{\mu}' = \hat{\mu} \text{ and } \hat{\lambda}' - \hat{\mu}' = 2\hat{\lambda} - (2n-1)\hat{\mu}.$$

Note that $\gamma_{x\lambda+y\mu} = \gamma_{-x\lambda-y\mu}$. Applying the diagram (5.1.9) with $\psi^{-}_{+,*}$ and $\psi^{+}_{-,*}$ switched to

$$\widehat{\Gamma}_{\mu}(\hat{\mu}') = \gamma_{\hat{\mu}'} = \widehat{\Gamma}_n, \widehat{\Gamma}_{-1}(\hat{\mu}') = \gamma_{\hat{\lambda}'+\hat{\mu}'} = \widehat{\Gamma}_{\mu}, \text{ and } \widehat{\Gamma}_0(\hat{\mu}') = \gamma_{\hat{\lambda}'} = \widehat{\Gamma}_{n-1},$$

we obtain the following commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{-1}(\hat{\mu}')) & \xrightarrow{\psi^{-1}_{+,0}(\hat{\mu}')} & \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{0}(\hat{\mu}')) \\ {\scriptstyle \psi^{\mu}_{-,-1}(\hat{\mu}')}\nwarrow & & \nearrow{\scriptstyle \psi^{\mu}_{-,0}(\hat{\mu}')} \\ & \underline{\mathrm{SHI}}(-\widehat{Y}(K), -\widehat{\Gamma}_{\mu}(\hat{\mu}')) & \end{array} \qquad (5.1.17)$$

where the notations $\hat{\mu}'$ in bypass maps indicate that they correspond to $\hat{\mu}'$. By comparing the grading shifts, we have

$$\psi^{-1}_{+,0}(\hat{\mu}') = \beta_- \text{ and } \psi^{\mu}_{-,-1}(\hat{\mu}') = \alpha_+.$$

Indeed, this can be obtained by a diagramatic way in Remark 5.1.11 and Remark 5.1.23.

Let $\delta : \widehat{\Gamma}^{i,+}_n \to \widehat{\Gamma}^{i,-}_{n-1}$ be the restriction of

$$\psi^{\mu}_{-,0}(\hat{\mu}') : \underline{\mathrm{SHI}}(-\widehat{Y}(K), -\widehat{\Gamma}_n) \to \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{n-1}).$$

Then (5.1.17) implies $\delta = \beta_- \circ \alpha_+ = \phi$.

Applying the negative bypass triangle in Theorem 5.1.10 to

$$\widehat{\Gamma}_{\mu}(\hat{\mu}') = \gamma_{\hat{\mu}'} = \widehat{\Gamma}_n, \widehat{\Gamma}_0(\hat{\mu}') = \gamma_{\hat{\lambda}'} = \widehat{\Gamma}_{n-1}, \text{ and } \widehat{\Gamma}_1(\hat{\mu}') = \gamma_{\hat{\lambda}'-\hat{\mu}'} = \gamma_{2\hat{\lambda}-(2n-1)\hat{\mu}},$$

we have the following exact triangle

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{0}(\hat{\mu}')) & \xrightarrow{\psi^{0}_{-,1}(\hat{\mu}')} & \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{1}(\hat{\mu}')) \\ {\scriptstyle \psi^{\mu}_{-,0}(\hat{\mu}')}\nwarrow & & \swarrow{\scriptstyle \psi^{1}_{-,\mu}(\hat{\mu}')} \\ & \underline{\mathrm{SHI}}(-\widehat{Y}(K), -\widehat{\Gamma}_{\mu}(\hat{\mu}')) & \end{array} \qquad (5.1.18)$$

By grading shifts in Theorem 5.1.10, the restriction of (5.1.18) on a single grading implies

$$H(\mathrm{Cone}(\delta)) \cong \underline{\mathrm{SHI}}(-\widehat{Y}(K), -\gamma_{2\hat{\lambda}-(2n-1)\hat{\mu}}, S, j_n) \qquad (5.1.19)$$

Then the isomorphism in (5.1.15) follows from (5.1.16) and (5.1.19).

Case 2. We do not suppose $\widehat{\Gamma}_k^{i,+} = \widehat{\Gamma}_k^{i,-} = 0$ for all $k \leq n-2$ in the diagram (5.1.11). Since $(n-1)q \geq 2g(K)$ and $i \in [\hat{i}^{\mu}_{min}, \hat{i}^{\mu}_{max}]$, we have

$$|\frac{i - \hat{i}^{\mu}_{min}}{q}|, |\frac{i - \hat{i}^{\mu}_{max}}{q}| \leq |\frac{\hat{i}^{\mu}_{max} - \hat{i}^{\mu}_{min}}{q}| = \frac{q - 1 + 2g(K)}{q} < n.$$

By (5.1.12) and (5.1.13), we have $\widehat{\Gamma}_0^{i,\pm} = 0$.

In this case, let

$$A'_s = (\bigoplus_{k \in \mathbb{Z} \backslash \{0\}} \underline{\mathrm{SHI}}(-Y \backslash K, -\widehat{\Gamma}_{\mu}, S, s + kq), d_s)$$

be the subcomplex of A_s. The quotient A_s/A'_s is $\underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{\mu}, S, s)$ with no differentials. Then we have a long exact sequence

$$\cdots \to H(A'_s) \to H(A_s) \to H(A_s/A'_s) \xrightarrow{\partial_*} H(A'_s)\{1\} \to \cdots$$

Since $\widehat{\Gamma}_0^{i,\pm} = 0$, by Theorem 2.2.6, we know that

$$H(A'_s) \cong \widehat{\Gamma}^{i,+}_{n-1} \oplus \widehat{\Gamma}^{i,-}_{n-1}. \tag{5.1.20}$$

It is straightforward to check $\partial_* = (\beta_+, \beta_-)$ under the isomorphism (5.1.20). Then by Case 1, we have

$$H(A_s) \cong H(\mathrm{Cone}(\partial_*)) \cong H(\mathrm{Cone}(f)) \cong H(\mathrm{Cone}(\phi)) \cong \underline{\mathrm{SHI}}(-\widehat{Y}(K), -\gamma_{2\hat{\lambda}-(2n-1)\hat{\mu}}, S, j_n).$$

□

Then we prove the large surgery formula for negative surgeres.

Theorem 5.1.29 (Theorem 1.3.10, $n > 0$)**.** *Suppose $\widehat{\mu} = q\mu + p\lambda$ with $q \in \mathbb{N}_+$ and suppose $\widehat{\lambda} = q_0\mu + p_0\lambda$ is defined as in Definition 5.1.1. Note that when $(q,p) = (1,0)$, we have $(q_0, p_0) = (0,1)$. For a fixed integer n satisfying $(n-1)q \geq 2g(K)$, suppose*

$$\hat{\mu}' = n\hat{\mu} - \hat{\lambda} = (nq - q_0)\mu + (np - p_0)\lambda.$$

For any integer s', suppose $[s']$ is the image of s' in $\mathbb{Z}_{(nq-q_0)}$. Suppose

$$s_{min} = -(nq - q_0 - 1) - (-\frac{q-1}{2}) + g(K) \text{ and } s_{max} = (nq - q_0 - 1) - (\frac{q-1}{2}) - g(K)$$

and suppose a (half) integer $s \in [s_{min}, s_{max}]$. For such n and s, there is an isomorphism

$$H(A_{-s}) \cong I^{\sharp}(-\widehat{Y}_{\hat{\mu}'}, [s - s_{min}]).$$

Remark 5.1.30. When $(n-1)q \geq 2g(K)$, there are more than $(nq - q_0)$ integers in the interval $[s_{min}, s_{max}]$. Thus, the bent complexes contain all information of $I^{\sharp}(-\widehat{Y}_{\hat{\mu}'})$.

Proof of Theorem 5.1.29. Since $(n-1)q \geq 2g(K)$, we apply Theorem 5.1.27 to obtain

$$H(A_{-s}) \cong \underline{\mathrm{SHI}}(-Y\backslash K, -\gamma_{2\hat{\lambda}-(2n-1)\hat{\mu}}, S, j_n - s).$$

We adapt the notations

$$\hat{\lambda}' = \hat{\lambda} - (n-1)\hat{\mu} \text{ and } \hat{\lambda}' - \hat{\mu}' = 2\hat{\lambda} - (2n-1)\hat{\mu} = (2q_0 - (2n-1)q)\mu + (2p_0 - (2n-1)p)\lambda$$

from the proof of Theorem 5.1.27. Then $\widehat{\Gamma}_1(\hat{\mu}') = \gamma_{2\hat{\lambda}-(2n-1)\hat{\mu}}$. Since $(n-1)q \geq 2g(K)$, we have

$$(2n-1)q - 2q_0 \geq nq - q_0 + 2g(K).$$

Hence we can apply Theorem 5.1.15 to obtain

$$I^{\sharp}(-\widehat{Y}_{\hat{\mu}'}, [s]) \cong \underline{\mathrm{SHI}}(-Y\backslash K, -\gamma_{2\hat{\lambda}-(2n-1)\hat{\mu}}, S, \hat{i}^{\sharp}_{max} - 2g(K) - s).$$

By direct calculation, we have

$$\begin{aligned} j_n - s_{min} =& \hat{i}^{\sharp}_{max} - \hat{i}^{n}_{max} + \hat{i}^{n}_{min} - \hat{i}^{\mu}_{min} - s_{min} \\ =& \hat{i}^{\sharp}_{max} - 2g(K) - (nq - q_0 - 1) - (-\frac{q-1}{2}) + g(K) - s_{min} \\ =& \hat{i}^{\sharp}_{max} - 2g(K). \end{aligned}$$

For any $s \in [s_{min}, s_{max}]$, we have

$$\begin{aligned} j_n - s =& \hat{i}^{\sharp}_{min} - \hat{i}^{n}_{min} + \hat{i}^{n}_{max} - \hat{i}^{\mu}_{max} - s \\ =& \hat{i}^{\sharp}_{min} + 2g(K) + (nq - q_0 - 1) - (\frac{q-1}{2}) - g(K) - s \\ \geq& \hat{i}^{\sharp}_{min} + 2g(K). \end{aligned}$$

Thus, the isomorphism follows from Definition 5.1.16 and Lemma 5.1.13. □

Finally, we state an instanton analog of [OS08b, Theorem 2.3] and [OS11, Theorem 4.1], which is an important step of the proof of the mapping cone formula.

Construction 5.1.31. Following notations in Construction 5.1.25. For $\circ \in \{+,-\}$, define

$$B_s^\circ = B_s^\circ(-\widehat{Y},\widehat{K}) := (\bigoplus_{k\in\mathbb{Z}} \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_\mu, S, s+kq), d_\circ)$$

and define

$$\pi_s^\circ : A_s \to B_s^\circ$$

by

$$\pi_s^+(x) = \begin{cases} x & k > 0, \\ 0 & k \le 0, \end{cases} \text{ and } \pi_s^-(x) = \begin{cases} 0 & k \ge 0, \\ 0 & k < 0, \end{cases}$$

where $x \in \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_\mu, S, s+kq)$.

Suppose $\hat{\mu} = q\mu + p\lambda$ with $q \in \mathbb{N}_+$. For n and s in Theorem 5.1.27, let $H(A_s), H(B_s^+), H(B_s^-)$ be homologies of complexes in Construction 5.1.25 and let $(\pi_s^+)_*, (\pi_s^-)_*$ denote the induced maps on homologies. Let j_n be the integer in Theorem 5.1.27 and write $\widehat{\Gamma}^{s,\sharp}$ for

$$\underline{\mathrm{SHI}}(-Y\backslash K, -\gamma_{2\hat{\lambda}-(2n-1)\hat{\mu}}, S, j_n + s).$$

By Theorem 5.1.27, we have an isomorphism

$$a_{s,n} : H(A_s) \xrightarrow{\cong} \widehat{\Gamma}^{s,\sharp}$$

We use notations in (5.1.11) and set $i = s$. Let

$$\rho_+ : \widehat{\Gamma}^{s,\sharp} \to \widehat{\Gamma}_n^{s,+}$$

be the restriction of $\psi_{-,\mu}^1(\hat{\mu}')$ in the proof of Theorem 5.1.27. Choose l as in the proof of Theorem 5.1.24 so that $\widehat{\Gamma}_{n+l}^{s,+} \subset G_+$. Note that $H(B_s^\pm) = \widehat{\Gamma}_{n+l}^{s,\pm}$ by the proof of Theorem 5.1.24. Let

$$\Psi_{+,n+l}^n : \widehat{\Gamma}_n^{s,+} \to \widehat{\Gamma}_{n+l}^{s,+}$$

be the composition of $\psi_{+,n+k+1}^{n+k}$ for $k = 0, \ldots, l-1$. Similarly, let

$$\rho_- : \widehat{\Gamma}^{s,\sharp} \to \widehat{\Gamma}_n^{s,-}$$

be the restriction of $\psi_{+,\mu}^1(\hat{\mu}')$ and let

$$\Psi_{-,n+l}^n : \widehat{\Gamma}_n^{s,-} \to \widehat{\Gamma}_{n+l}^{s,-} \subset G_-$$

be the composition of $\psi^{n+k}_{-,n+k+1}$ for $k=0,\dots,l-1$.

Proposition 5.1.32. *The following diagram commutes*

$$\begin{CD} H(A_s) @>{(\pi_s^{\pm})_*}>> H(B_s^{\pm}) \\ @V{a_{s,n}}VV @VV{=}V \\ \widehat{\Gamma}^{s,\sharp} @>{\Psi^n_{\pm,n+l}\circ\rho_{\pm}}>> \widehat{\Gamma}^{s,\pm}_{n+l}, \end{CD}$$

Proof. The proof is straightforward by the proof of Theorem 5.1.27. □

Remark 5.1.33. By direct calculation, the difference of gradings of $\widehat{\Gamma}^{s,+}_{n+l}$ and $\widehat{\Gamma}^{s,-}_{n+l}$ is

$$\begin{aligned}(\hat{i}^{n+l}_{min}-\hat{i}^{n}_{min}+\hat{i}^{n}_{max}-\hat{i}^{\mu}_{max})&-(\hat{i}^{n+l}_{max}-\hat{i}^{n}_{max}+\hat{i}^{n}_{min}-\hat{i}^{\mu}_{min})\\ &=-(\hat{i}^{n+l}_{max}-\hat{i}^{n+l}_{min})+2(\hat{i}^{n}_{max}-\hat{i}^{n}_{min})-(\hat{i}^{\mu}_{max}-\hat{i}^{\mu}_{min})\\ &=-(n+l)q+q_0+2(nq-q_0)-q\\ &=(n-l-1)q-q_0.\end{aligned}$$

By Lemma 5.1.13, the space $\widehat{\Gamma}^{s,+}_{n+l}$ and $\widehat{\Gamma}^{s,-}_{n+l}$ correspond to $I^{\sharp}(-\widehat{Y},[s_0-q_0])$ and $I^{\sharp}(-\widehat{Y},[s_0])$ for some integer s_0, respectively. Note that the core knot corresponding to $\hat{\mu}=q\mu+p\lambda$ is isotopic to the curve $q_0\mu+p_0\lambda$ on $\partial Y\backslash K$.

5.1.6 Dual bent complexes

In this subsection, we construct the dual bent complex and relate its homology to large positive surgeries. Proofs are similar to those in Subsection 5.1.5, so we only point out the difference.

Construction 5.1.34. Following notations in Construction 5.1.25. For any integer s, define the **dual bent complex** as

$$A_s^{\vee}=A_s^{\vee}(-\widehat{Y},\widehat{K}):=(\bigoplus_{k\in\mathbb{Z}}\underline{\mathrm{SHI}}(-Y\backslash K,-\widehat{\Gamma}_{\mu},S,s+kq),d_s^{\vee}),$$

where for any element $x\in\underline{\mathrm{SHI}}(-Y\backslash K,-\widehat{\Gamma}_{\mu},S,s+kq)$,

$$d_s^{\vee}(x)=\begin{cases}d_-(x) & k>0,\\ d_+(x)+d_-(x) & k=0,\\ d_+(x) & k<0.\end{cases}$$

Similar to Theorem 5.1.27 and Theorem 5.1.29, we have the following theorems.

Theorem 5.1.35. *Suppose $\hat{\mu} = q\mu + p\lambda$ with $q \in \mathbb{N}_+$. For any integer s, let $H(A_s^\vee)$ denote the homology of the bent complex $A_s^\vee$ in Construction 5.1.34. For any integer n satisfying $(n-1)q \geq 2g(K)$, we have an isomorphism for some integer $j_n^\vee$:*

$$a_{s,n}^\vee : H(A_s^\vee) \xrightarrow{\cong} \underline{\mathrm{SHI}}(-Y\backslash K, -\gamma_{2\hat{\lambda}+(2n+1)\hat{\mu}}, S, s + j_n^\vee). \tag{5.1.21}$$

Suppose the maximal and minimal nontrivial gradings of $\underline{\mathrm{SHI}}(-Y\backslash K, -\gamma_{2\hat{\lambda}+(2n+1)\hat{\mu}})$ are $\hat{i}_{max}^{\sharp,\vee}$ and $\hat{i}_{min}^{\sharp,\vee}$, which can be calculated by Lemma 5.1.8. Then we have

$$j_n^\vee = \hat{i}_{max}^{\sharp,\vee} - \hat{i}_{max}^{-n} + \hat{i}_{min}^{-n} - \hat{i}_{min}^{\mu} = \hat{i}_{min}^{\sharp,\vee} - \hat{i}_{min}^{-n} + \hat{i}_{max}^{-n} - \hat{i}_{max}^{\mu}.$$

Theorem 5.1.36 (Theorem 1.3.10, $n < 0$)**.** *Suppose $\widehat{\mu} = q\mu + p\lambda$ with $q \in \mathbb{N}_+$ and suppose $\widehat{\lambda} = q_0\mu + p_0\lambda$ is defined as in Definition 5.1.1. Note that when $(q, p) = (1, 0)$, we have $(q_0, p_0) = (0, 1)$. For a fixed integer n satisfying $(n-1)q \geq 2g(K)$, suppose*

$$\hat{\mu}'' = n\hat{\mu} + \hat{\lambda} = (nq + q_0)\mu + (np + p_0)\lambda.$$

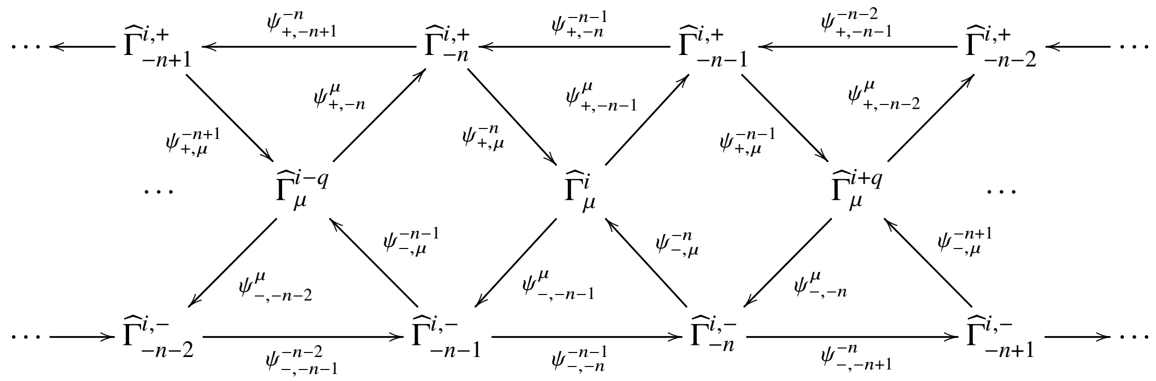
For any integer s', suppose $[s']$ is the image of s' in $\mathbb{Z}_{(nq+q_0)}$. Suppose

$$s_{min}^\vee = -(nq + q_0 - 1) - (-\frac{q-1}{2}) + g(K) \text{ and } s_{max}^\vee = (nq + q_0 - 1) - (\frac{q-1}{2}) - g(K)$$

and suppose a (half) integer $s \in [s_{min}^\vee, s_{max}^\vee]$. For such n and s, there is an isomorphism

$$H(A_{-s}^\vee) \cong I^\sharp(-\widehat{Y}_{\hat{\mu}''}, [s - s_{min}^\vee]).$$

Proof of Theorem 5.1.35. Instead of using the diagram 5.1.11, we use the following diagram of exact triangles from Proposition 5.1.10:



(5.1.22)

where we write

$$\begin{aligned}
\widehat{\Gamma}^{i}_{\mu} &= \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{\mu}, S, i)\\
\widehat{\Gamma}^{i,+}_{-k} &= \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{-k}, S, i+\hat{i}^{-k}_{max}-\hat{i}^{-n}_{max}+\hat{i}^{-n}_{min}-\hat{i}^{\mu}_{min})\\
\widehat{\Gamma}^{i,-}_{-k} &= \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{-k}, S, i+\hat{i}^{-k}_{min}-\hat{i}^{-n}_{min}+\hat{i}^{-n}_{max}-\hat{i}^{\mu}_{max})
\end{aligned}$$

for any $k \in \mathbb{N}$, and we abuse notation so that the maps $\psi^{*}_{+,*}, \psi^{*}_{-,*}$ also denote the restrictions on corresponding gradings. In this case, we have

$$\widehat{\Gamma}^{i,+}_{-n-k} \cong \widehat{\Gamma}^{i,+}_{-n-k-1} \text{ for } k > \frac{\hat{i}^{\mu}_{max}-i}{q} \text{ and } \widehat{\Gamma}^{i,+}_{-n+k} = 0 \text{ for } -k < \frac{\hat{i}^{\mu}_{min}-i}{q}, \tag{5.1.23}$$

$$\widehat{\Gamma}^{i,-}_{-n-k} \cong \widehat{\Gamma}^{i,-}_{-n-k-1} \text{ for } k > \frac{i-\hat{i}^{\mu}_{min}}{q} \text{ and } \widehat{\Gamma}^{i,-}_{-n+k} = 0 \text{ for } -k < \frac{i-\hat{i}^{\mu}_{max}}{q}. \tag{5.1.24}$$

By Proposition 2.2.5 and Theorem 2.2.6, there exist spectral sequences from

$$\bigoplus_{k\in\mathbb{Z}} \widehat{\Gamma}^{i+kq}_{\mu}$$

to $\widehat{\Gamma}^{i,+}_{-n-l}$ and $\widehat{\Gamma}^{i,-}_{-n-l}$ for some large l. By Lemma 5.1.19, those spectral sequences are isomorphic to $\{(E_{r,+}, d_{r,+})\}_{r\geq 1}$ and $\{(E_{r,-}, d_{r,-})\}_{r\geq 1}$ in Theorem 5.1.24, hence we can define the dual bent complex by maps in (5.1.22).

By Definition 5.1.2, we set

$$\hat{\mu}'' = n\hat{\mu} + \hat{\lambda} \text{ and } \hat{\lambda}'' = -\hat{\mu}.$$

Then

$$\hat{\lambda}''-\hat{\mu}''=-\hat{\lambda}-(n+1)\hat{\mu} \text{ and } \hat{\lambda}''-2\hat{\mu}''=-2\hat{\lambda}-(2n+1)\hat{\mu}.$$

Note that $\gamma_{x\lambda+y\mu}=\gamma_{-x\lambda-y\mu}$.

Similar to the proof of Theorem 5.1.27, we consider two cases and finally obtain that

$$\begin{aligned}H(A_i^\vee)\cong&H(\mathrm{Cone}(\psi^{-n}_{+,\mu}+\psi^{-n}_{-,\mu}:\widehat{\Gamma}^{i,+}_{-n}\oplus\widehat{\Gamma}^{i,-}_{-n}\to\widehat{\Gamma}^{i}_{\mu})\\ \cong&H(\mathrm{Cone}(\psi^{\mu}_{-,-n-1}\circ\psi^{-n}_{+,\mu}:\widehat{\Gamma}^{i,+}_{-n}\to\widehat{\Gamma}^{i,-}_{-n-1}))\\ \cong&\underline{\mathrm{SHI}}(-Y\backslash K,-\gamma_{2\hat{\lambda}+(2n+1)\hat{\mu}},S,i+j_n^\vee).\end{aligned}$$

□

Proof of Theorem 5.1.36. Similar to the proof of Theorem 5.1.29, the isomorphism follows from Theorem 5.1.15, Definition 5.1.16, and Lemma 5.1.13. □

The following proposition explains the name of the 'dual bent complex'.

Proposition 5.1.37. $A_s^\vee(-\widehat{Y},\widehat{K})$ *is the dual complex of* $A_{-s}(\widehat{Y},\widehat{K})$.

Proof. Suppose $(\bar{Y},\bar{K})=(-Y,K)$ is the mirror knot of (Y,K). Note that $(-\bar{Y},\bar{K})=(Y,K)$. Suppose S is the Seifert surface of S of K. Then $-S$ is the induced Seifert surface of $\bar{K}$. By Theorem 2.3.20, we have canonical isomorphisms

$$\begin{aligned}\underline{\mathrm{SHI}}(-\bar{Y}(\bar{K}),-\widehat{\Gamma}_n,-S,i)=&\underline{\mathrm{SHI}}(Y\backslash K,-\widehat{\Gamma}_{-n},-S,i)\\ \cong&\underline{\mathrm{SHI}}(Y\backslash K,-\widehat{\Gamma}_{-n},S,-i)\\ \cong&\mathrm{Hom}_{\mathbb{C}}(\underline{\mathrm{SHI}}(-Y\backslash K,-\widehat{\Gamma}_{-n},S,-i),\mathbb{C})\end{aligned}.$$

Then this proposition follows from the fact that both diagram (5.1.11) and diagram (5.1.22) can be used to define the bent complex and the dual bent complex. □

5.1.7 Grading shifts of differentials

In this subsection, we study the grading shifts of differentials d_+ and d_+ and relate the bent complex to the dual bent complex. First, it is straightforward to check from the construction that the map d_+ increases the $\mathbb{Z}$-grading and d_- decreases the $\mathbb{Z}$-grading. So we focus on the grading shifts of d_+ and d_- on the relative $\mathbb{Z}_2$-grading.

Convention. Throughout this subsection, 'grading' means the relative $\mathbb{Z}_2$-grading and we set $M=Y\backslash K$ for a rationally null-homologous knot $K\subset Y$. The bypass map $\psi^*_{+,*}$ and the corresponding negative one $\psi^*_{-,*}$ are from $\underline{\mathrm{SHI}}(-M,-\gamma_1)$ to $\underline{\mathrm{SHI}}(-M,-\gamma_2)$ for some γ_1 and γ_2 consisting of two parallel simple closed curves.

Since all bypass maps are homogeneous (they are constructed by cobordism maps, *c.f.* the proof of [BS22, Theorem 1.20]), the differentials d_+ and d_- are also homogeneous. To study the grading shifts of d_+ and d_-, we first study the isomorphism

$$\iota_\gamma : \underline{\mathrm{SHI}}(-M,-\gamma) \xrightarrow{\cong} \underline{\mathrm{SHI}}(-M,\gamma) \xrightarrow{=} \underline{\mathrm{SHI}}(-M,-\gamma) \tag{5.1.25}$$

defined in (5.1.5) more carefully.

By construction of $\underline{\mathrm{SHI}}(-M,-\gamma)$ in [KM10b, BS15], we can construct a closure (Y',R,ω) of $(-M,-\gamma)$ with $g(R)\geq 2$ and take the $(2,2g(R)-2)$-eigenspace of $(\mu(\mathrm{pt}),\mu(R))$ on $I^\omega(Y')$. It is straightforward to check that $(Y',-R,\omega)$ is a closure of $(-M,\gamma)$. Hence we can define $\underline{\mathrm{SHI}}(-M,\gamma)$ by the $(2,2g(R)-2)$-eigenspace of $(\mu(\mathrm{pt}),\mu(-R))$ on $I^\omega(Y')$, which is the same as the $(2,2-2g(R))$-eigenspace of $(\mu(\mathrm{pt}),\mu(R))$ on $I^\omega(Y')$. Note that $I^\omega(Y')$ has a $\mathbb{Z}_8$-grading and $\mu(\mathrm{pt})$ and $\mu(R)$ have degree -4 and -2, respectively. The canonical isomorphism $\underline{\mathrm{SHI}}(-M,-\gamma)\cong\underline{\mathrm{SHI}}(-M,\gamma)$ in (5.1.25) comes from the map sending

$$(v_0,v_1,v_2,v_3,v_4,v_5,v_6,v_7)\in I^\omega(Y')$$

to

$$(v_0,v_1,-v_2,-v_3,v_4,v_5,-v_6,-v_7),$$

which preserves the $\mathbb{Z}_2$-grading induced by the $\mathbb{Z}_8$-grading.

Since γ and $-\gamma$ are isotopic on $\partial M\cong T^2$, there is an identification $\underline{\mathrm{SHI}}(-M,-\gamma) = \underline{\mathrm{SHI}}(-M,\gamma)$. However, this identification may depend on the isotopy since there may be some basepoint moving map similar to Heegaard Floer theory [Sar15, Zem17]. Since we do not care about the precise identification, we omit discussion about specifying the isotopy. .

Lemma 5.1.38. *Suppose $\psi^*_{+,*}$ and $\psi^*_{-,*}$ are two bypass maps from $\underline{\mathrm{SHI}}(-M,-\gamma_1)$ to $\underline{\mathrm{SHI}}(-M,-\gamma_2)$ and suppose ι_{γ_1} and ι_{γ_2} are isomorphisms defined in (5.1.5). Under some choices of isotopies of sutures, we have*

$$\psi^*_{-,*}\circ\iota_{\gamma_1}=\iota_{\gamma_2}\circ\psi^*_{-,*}.$$

Proof. By construction in Subsection 5.1.2, the bypass arc related to $\psi^*_{+,*}$ on $(Y\backslash K,\gamma_{x\lambda+y\mu})$ is the same as the bypass arc related to $\psi^*_{-,*}$ on $(Y\backslash K,-\gamma_{x\lambda+y\mu})$. The lemma follows from the construction of the isomorphism ι_γ. □

Corollary 5.1.39. *The isomorphism ι_γ induces an isomorphism between spectral sequences $\{(E_{r,+},d_{r,+})\}_{r\geq1}$ and $\{(E_{r,-},d_{r,-})\}_{r\geq1}$ constructed in Theorem 5.1.24 and hence induces an isomorphism between the chain complexes*

$$(\underline{\mathrm{KHI}}(-Y,K),d_+) \text{ and } (\underline{\mathrm{KHI}}(-Y,K),d_-).$$

Moreover, it induces a canonical identification between A_{-s} and A_s.

Lemma 5.1.40. *Suppose $\psi^*_{+,*}$ and $\psi^*_{-,*}$ are two bypass maps from $\underline{\mathrm{SHI}}(-M,-\gamma_1)$ to $\underline{\mathrm{SHI}}(-M,-\gamma_2)$. If x is a homogeneous element in $\underline{\mathrm{SHI}}(-M,-\gamma_1)$, then $\psi^*_{+,*}(x)$ and $\psi^*_{-,*}(x)$ are homogeneous elements in $\underline{\mathrm{SHI}}(-M,-\gamma_2)$ and they have the same grading.*

Proof. It follows from Lemma 5.1.38 and the fact that the isomorphism ι_γ preserves the grading for any $\gamma \subset \partial M$. □

Proposition 5.1.41. *Suppose d_+ and d_- are differentials on $\underline{\mathrm{KHI}}(-Y,K)$ induced by spectral sequences $\{(E_{r,+},d_{r,+})\}_{r\geq 1}$ and $\{(E_{r,-},d_{r,-})\}_{r\geq 1}$ in Theorem 5.1.24. For any homogeneous element $x \in \underline{\mathrm{KHI}}(-Y,K)$, the gradings of $d_+(x)$ and $d_-(x)$ are different from the grading of x.*

Proof. We only prove for $d_+(x)$. The proof for $d_-(x)$ is similar. We adapt notations in diagram (5.1.11). Without loss of generality, suppose $x \in \widehat{\Gamma}^i_\mu$. Consider the projection y of $d_+(x)$ on $\widehat{\Gamma}^{i+kq}_\mu$ for some $k \in \mathbb{N}_+$. By construction of d_+, there exist homogeneous elements $z \in \widehat{\Gamma}^{i,+}_{n-1}$ and $w \in \widehat{\Gamma}^{i,+}_{n-k}$ so that

$$y = \psi^{n-k}_{+,\mu}(w) \text{ and } z = \psi^{\mu}_{+,n-1}(x) = \psi^{n-2}_{+,n-1} \circ \cdots \circ \psi^{n-k}_{+,n-k+1}(w).$$

By Lemma 5.1.40, the element

$$z' := \psi^{n-2}_{-,n-1} \circ \cdots \circ \psi^{n-k}_{-,n-k+1}(w)$$

has the same grading as z. By Lemma 5.1.19, we have

$$\psi^{n-1}_{+,\mu}(z') = y.$$

Define

$$u := \psi^{n-1}_{+,n}(z') \text{ and } u' := \psi^{n-1}_{-,n}(z').$$

By Lemma 5.1.40, they have the same grading. By 5.1.19, we have

$$\psi^{n}_{+,\mu}(u') = y.$$

Let $\mathrm{gr}_2(x)$ denote the grading of x and let $\mathrm{gr}_2(\psi^*_{+,*})$ denote the grading shift of $\psi^*_{+,*}$. Then we have

$$\begin{aligned}\mathrm{gr}_2(y)-\mathrm{gr}_2(x) &= (\mathrm{gr}_2(y)-\mathrm{gr}_2(u'))+(\mathrm{gr}_2(u)-\mathrm{gr}_2(z'))+(\mathrm{gr}_2(z)-\mathrm{gr}_2(x))\\ &= \mathrm{gr}_2(\psi^n_{+,\mu})+\mathrm{gr}_2(\psi^{n-1}_{+,n})+\mathrm{gr}_2(\psi^{\mu}_{+,n-1})\\ &= 1,\end{aligned}$$

where the last equation follows from the fact that the bypass exact triangle shifts the grading (the bypass exact triangle comes from the surgery exact triangle, *c.f.* the proof of [BS22, Theorem 1.20]). Since any projection of $d_+(x)$ has different grading from x, the we know that $d_+(x)$ has different grading from x. □

5.2 Vanishing results about contact elements

In this section, we study contact elements in Heegaard Floer theory and instanton theory. We only need Corollary 5.2.19 in the rest sections.

5.2.1 Contact elements in Heegaard Floer theory

In this subsection, we review the strategy to prove the vanishing result about Giroux torsion by Ghiggini-Honda-Van Horn-Morris [GHVHM08].

Suppose (N,ξ) is a contact 3-manifold with convex boundary and dividing set Γ on ∂N. Honda-Kazez-Matić [HKM09] defined an element $c(N,\Gamma,\xi)$ in sutured Floer homology $SFH(-N,-\Gamma)$, called the **contact element** of (N,ξ). When (N,ξ) is obtained from a closed contact 3-manifold (Y,ξ') by removing a 3-ball, the element

$$c(N,\Gamma,\xi)\in SFH(-N,-\Gamma)\cong \widehat{HF}(-Y)$$

recovers the contact element $c(Y,\xi')\in\widehat{HF}(-Y)$ defined by Ozsváth-Szabó [OS05a].

Definition 5.2.1. A contact closed 3-manifold (Y,ξ) has **Giroux torsion** if there is an embedding of $(T^2\times[0,1],\eta_{2\pi})$ into (Y,ξ), where (x,y,t) are coordinates on $T^2\times[0,1]\cong \mathbb{R}^2/\mathbb{Z}^2\times[0,1]$ and

$$\eta_{2\pi}=\mathrm{Ker}(cos(2\pi t)dx-sin(2\pi t)dy).$$

We have the following vanishing result.

Theorem 5.2.2 ([GHVHM08, Theorem 1])**.** *If a closed contact 3-manifold (Y,ξ) has Giroux torsion, then its contact element $c(Y,\xi)\in\widehat{HF}(-Y)$ vanishes.*

Remark 5.2.3. The statement of Theorem 5.2.2 in [GHVHM08] is about $\mathbb{Z}$ coefficient. However, since the naturality of SFH is only proved for $\mathbb{F}_2$ coefficient [JTZ21], the contact element in $\mathbb{Z}$ coefficient is not well-defined. Some progress about the naturality for $\mathbb{Z}$ coefficient is made in [Gar19].

Remark 5.2.4. There are many partial results and applications of Theorem 5.2.2. See the introduction of [GHVHM08].

Following the notations in [Hon00, Section 5.2], consider a basic slice $N_0 = (T^2 \times I, \bar{\xi})$ with the dividing set Γ_* on $T^2 \times \{i\}$ for $i = 0, 1$ consisting of two parallel curves of slopes $s_0 = \infty$ and $s_1 = 0$. There are two possible choices of tight structures on N_0 corresponding to two bypasses $\psi^{\mu}_{+,0}$ and $\psi^{\mu}_{-,0}$. They are both positively co-oriented but have different orientations. Hence the relative Euler classes differ by signs. Let $\bar{\xi}$ be the tight structure on N_0 corresponding to $\psi^{\mu}_{+,0}$. Let $N_{\frac{n\pi}{2}}$ be obtained from N_0 by rotating counterclockwise by $\frac{n\pi}{2}$. Note that N_π is the basic slice corresponding to $\psi^{\mu}_{-,0}$ and $N_{\frac{n\pi}{2}+2\pi} = N_{\frac{n\pi}{2}}$. Define

$$(N_*, \zeta_1^+) = N_0 \cup N_{\frac{\pi}{2}} \cup N_\pi \cup N_{\frac{3\pi}{2}} \cup N_{2\pi} \text{ and } (N_*, \zeta_1^-) = N_\pi \cup N_{\frac{3\pi}{2}} \cup N_{2\pi} \cup N_{\frac{5\pi}{2}} \cup N_{3\pi}.$$

Then Theorem 5.2.2 follows from the following three lemmas.

Lemma 5.2.5 ([GHVHM08, Lemma 5]). *A contact closed 3-manifold (Y, ξ) has Giroux torsion if and only if there exists an embedding of $(N_*, \Gamma_*, \zeta_1^+)$ or $(N_*, \Gamma_*, \zeta_1^+)$ into (Y, ξ).*

Remark 5.2.6. In the definition of Giroux torsion, there is no condition on the orientation of the contact structure. By construction, the contact structures ζ_1^+ and ζ_1^- differ by orientations. In [GHVHM08], the authors did not deal with these two contact structures separately (*c.f.* the definition of ζ_0 in [GHVHM08]) since the proofs are almost identical. Also, in the original statement of [GHVHM08, Lemma 5], the slopes of dividing set on ∂N_* are -1 and -2, respectively. However, there is a diffeomorphism of $T^2 \times I$ sending the slopes to ∞ and 0, respectively. Note that under this diffeomorphism, the slope ∞ is sent to -1.

Lemma 5.2.7 ([HKM09, Theorem 4.5]). *Let (Y, ξ) be a closed contact 3-manifold and $N \subset Y$ be a compact submanifold (without any closed components) with convex boundary and dividing set Γ. If $c(N, \Gamma, \xi|_N) = 0$, then $c(Y, \xi) = 0$.*

Lemma 5.2.8 (From the proof of [GHVHM08, Theorem 1]). *The elements $c(N_*, \Gamma_*, \zeta_1^+)$ and $c(N_*, \Gamma_*, \zeta_1^-)$ vanish.*

5.2.2 Construction of instanton contact elements

In [BS16b], Baldwin-Sivek constructed a contact invariant in sutured instanton theory which we call the **instanton contact element**. In this subsection, we review the construction and prove the following theorem.

Theorem 5.2.9. *Suppose (N,ξ) is a contact 3-manifold with convex boundary and dividing set Γ on ∂N. Suppose S is an admissible surface (c.f. Definition 2.3.19) in (N,Γ) and suppose S_+ and S_- are positive region and negative region of S with respect to ξ, respectively. We write the $\mathbb{Z}$-grading associated to S as*

$$\underline{\mathrm{SHI}}(-N,-\Gamma)=\bigoplus_{i\in\mathbb{Z}}\underline{\mathrm{SHI}}(-N,-\Gamma,S,i).$$

Then the instanton contact element $\theta(N,\Gamma,\xi)$ lives in

$$\underline{\mathrm{SHI}}(-N,-\Gamma,S,\frac{\chi(S_+)-\chi(S_-)}{2}).$$

Definition 5.2.10. Suppose (M,γ) is a balanced sutured manifold. A contact structure ξ on M is said to be **compatible** if ∂M is convex and γ is the dividing set on ∂M.

For a balanced sutured manifold (M,γ) and a compatible contact structure ξ, there are a few ways to decompose ξ [HKM09, BS16b].

Partial open book decomposition. A partial open book decomposition is a triple (S,P,h) where S is a compact surface with non-empty boundary, $P\subset S$ a subsurface, and $h:P\to S$ an embedding so that h is the identity on $\partial P\cap\partial S$.

Contact cellular decomposition. A contact cellular decomposition of ξ over (M,γ) is, roughly speaking, a Legendrian graph $\mathcal{K}\subset M$ so that $\partial\mathcal{K}\subset\gamma$ and $M\backslash\mathrm{int}N(\mathcal{K})$ is diffeomorphic to a product $[-1,1]\times F$ for some surface F withboundary and ξ restricts to the $[-1,1]$-invariant contact structure on $M\backslash\mathrm{int}N(\mathcal{K})\cong[-1,1]\times F$.

Contact handle decomposition. A contact handle decomposition is a decomposition of (M,γ,ξ) into contact 0-, 1-, and 2-handles described above.

These three decompositions can be related to each other as follows.

Suppose we have a contact cellular decomposition, *i.e.*, a Legendrian graph $\mathcal{K}\subset M$ so that $M\backslash\mathrm{int}N(\mathcal{K})$ is a product manifold equipped with the product contact structure. Then $M\backslash\mathrm{int}N(\mathcal{K})$ equipped with the restriction of ξ can be decomposed into a contact 0-handle and a few contact 1-handles. Furthermore, each edge of the Legendrian graph $\mathcal{K}$ corresponds to a contact 2-handle attached along a meridian of the edge. This gives rise to a contact handle decomposition of (M,γ,ξ).

Suppose we have a contact handle decomposition of (M,γ,ξ), we can obtain a partial open book decomposition as follows. All 0- and 1- handle form a product sutured manifold $([-1,1]\times S,\{0\}\times\partial S)$. Suppose 2-handles are attached along curves δ_1,..., δ_n. Let $P\subset\{1\}\times S$ be a neighborhood of $(\delta_1\cup\cdots\cup\delta_n)\cap\{1\}\times S$. Isotope $(\delta_1\cup\cdots\cup\delta_n)\cap\{-1\}\times S$ through $[-1,1]\times S$ onto $\{1\}\times S$. Let $h:P\to S$ be the embedding so that $h|_{\partial S\cap\partial P}$ is the identity and $\delta_i\cap\{1\}\times S$ is sent to the image of $\delta_i\times\{-1\}\times S$ under the isotopy for $i=1,\dots,n$. Then (S,P,h) is a partial open book decomposition of (M,γ,ξ).

Suppose we have a partial open book decomposition (S,P,h) of (M,γ,ξ). We know that $([-1,1]\times S,\{0\}\times\partial S)$ is a product sutured manifold that admits a product contact structure ξ_0. This can be decomposed into a contact 0-handle and a few contact 1-handles. Let $a_1,\dots,a_n$ be a collection of disjoint properly embedded arcs on S so that $a_i\subset P$ and $S-(a_1\cup\cdots\cup a_n)$ retracts to $S-P$. Let δ_i be the union of a_i and $h(a_i)$. Then (M,γ,ξ) is obtained from $([-1,1]\times S,\{0\}\times\partial S,\xi_0)$ by attaching contact 2-handles along all δ_i.

Definition 5.2.11 ([BS16b]). Suppose (M,γ) is a balanced sutured manifold and ξ is a compatible contact structure. Suppose ξ has a partial open book decomposition (S,h,P). Let δ_1,..., δ_n be the attaching curves of the contact 2-handles so that (M,γ,ξ) is obtained from $([-1,1]\times S,\{0\}\times\partial S)$ as above. Suppose the element $\mathbf{1}$ is the generator of

$$\underline{\mathrm{SHI}}(-[-1,1]\times S,-\{0\}\times\partial S)\cong\mathbb{C}.$$

Then the **instanton contact element** of (M,γ,ξ) is

$$\theta(M,\gamma,\xi):=C_{\delta_n}\circ\cdots\circ C_{\delta_1}(\mathbf{1})\in SHI(-M,-\gamma),$$

where C_{δ_i} is the contact gluing map associated to the contact 2-handle attachment along δ_i (*c.f.* Subsection 2.3.4).

Theorem 5.2.12 (Baldwin-Sivek [BS16b]). *Suppose (M,γ) is a balanced sutured manifold, and ξ is a compatible contact structure. Then the instanton contact element $\theta(M,\gamma,\xi)\in SHI(-M,-\gamma)$ is independent of the choice of the partial open book decomposition and is well-defined up to a unit. In particular, the non-vanishing of the instanton contact element is an invariant property for the contact structure.*

Then we prove the main theorem of this subsection.

Proof of Theorem 5.2.9. First, we prove the instanton contact element is homogeneous with respect to the $\mathbb{Z}$-grading of $SHI(-M,-\gamma)$ associated to S. From [HKM09, Theorem 1.1], any triple (M,γ,ξ) admits a contact cell decomposition. Hence there exists a Legendrian graph $\mathcal{K}$, so that $(M\backslash\mathrm{int}N(\mathcal{K}),\xi|_{M\backslash\mathrm{int}N(\mathcal{K})})$ is contactomorphic to $([-1,1]\times F,\xi_0)$ for some surface F with boundary and the product contact structure ξ_0. Let δ_1,..., δ_n be a set of meridians of K, one for each edge of $\mathcal{K}$. Then we can obtain the original ξ on M from $([-1,1]\times F,\xi_0)$ by attaching contact 2-handles along $\delta_1,\dots,\delta_n$. As discussed above, this gives rise to a contact handle decomposition and hence a partial open book decomposition. From Definition 5.2.11, we know that

$$\theta(M,\gamma,\xi)=C_{\delta_n}\circ\cdots\circ C_{\delta_1}(\mathbf{1})\in\underline{\mathrm{SHI}}(-M,-\gamma),$$

where C_{δ_i} is the contact gluing map associated to the contact 2-handle attachment along δ_i.

Suppose $S\subset(M,\gamma)$ is an admissible surface. We can isotope S so that it intersects $\mathcal{K}$ transversely and disjoint from all δ_i. Write

$$S_{\mathcal{K}}=S\cap(M\backslash\mathrm{int}N(\mathcal{K})).$$

We can consider it as a surface inside the product sutured manifold $([-1,1]\times S,\{0\}\times\partial S)$. Note that $\partial S_{\mathcal{K}}\backslash\partial S$ are all meridians of $\mathcal{K}$ and, by construction, each meridian of $\mathcal{K}$ has two intersections with the dividing set on $\partial(M\backslash\mathrm{int}N(\mathcal{K}))$, which is also identified with

$$\{1\}\times\partial S\subset[-1,1]\times\{1\}\times S.$$

So $S_{\mathcal{K}}$ is also admissible inside $([-1,1]\times S,\{0\}\times\partial S)$. Since

$$\underline{\mathrm{SHI}}(-[-1,1]\times S,-\{0\}\times\partial S)\cong\mathbb{C},$$

we know that there exists $i_0\in\mathbb{Z}$ so that

$$\mathbf{1}\in\underline{\mathrm{SHI}}(-[-1,1]\times S,-\{0\}\times\partial S,S_{\mathcal{K}},i_0).$$

From Proposition 4.1.6, we know that all maps C_{δ_i} preserve the gradings associated to $S_{\mathcal{K}}$ and S, respectively. Thus, we conclude that

$$\theta(M,\gamma,\xi)=C_{\delta_n}\circ\cdots\circ C_{\delta_1}(\mathbf{1})\in\underline{\mathrm{SHI}}(-M,-\gamma,i_0).$$

Then we need to figure out i_0. Since $\underline{\mathrm{SHI}}(-[-1,1]\times S,-\{0\}\times\partial S)$ is one-dimensional, the integer i_0 is determined by its graded Euler characteristic (we fix the closure to resolve the ambiguity of $\pm H$). By results in Section 4.1, it suffices to calculate i_0 when replacing $\underline{\mathrm{SHI}}$ by SFH. Note that the contact element of any contact structure ξ compatible with (M,γ)

lives in $SFH(-M,-\gamma,\mathfrak{s}_\xi)$, where $\mathfrak{s}_\xi$ is the relative spin^c structure corresponding to ξ. The formula of i_0 then follows from [Hon00, Proposition 4.5]. □

5.2.3 Vanishing results about Giroux torsion

Instanton contact elements share similar properties with the contact elements in SFH. In this subsection, we prove the following theorem.

Theorem 5.2.13. *If a closed contact 3-manifold (Y,ξ) has Giroux torsion, then its instanton contact element $\theta(Y,\xi)\in I^\sharp(-Y)$ vanishes.*

First, we need to prove lemmas similar to Lemma 5.2.7 and Lemma 5.2.8.

The analog of Lemma 5.2.7 follows directly from the following proposition.

Proposition 5.2.14 ([Li18, Corollary 1.4], see also [BS16b, Theorem 1.2])**.** *Consider the notations as above. If the contact structure ξ on $M'\backslash \mathrm{int}M$ is a restriction of a contact structure ξ' on M', then we have*

$$\Phi_\xi(\theta(M,\gamma,\xi'|_M))=\theta(M',\gamma',\xi')\in \underline{\mathrm{SHI}}(-M',-\gamma').$$

Corollary 5.2.15. *Let (Y,ξ) be a closed contact 3-manifold and $N\subset Y$ be a compact submanifold (without any closed components) with convex boundary and dividing set Γ. If $\theta(N,\Gamma,\xi|_N)=0$, then $\theta(Y,\xi)=0$.*

The following proposition is the analog of Lemma 5.2.8.

Proposition 5.2.16. *The instanton contact elements $\theta(N_*,\Gamma_*,\zeta_1^+)$ and $\theta(N_*,\Gamma_*,\zeta_1^-)$ vanish.*

Proof. Since instanton contact elements share most properties with contact elements, we can apply the proof of Lemma 5.2.8 with mild changes. We sketch the proof and point out the main difference. For simplicity, we only consider $\theta(N_*,\Gamma_*,\zeta_1^+)$. The proof for $\theta(N_*,\Gamma_*,\zeta_1^-)$ is almost identical.

Take a copy $T_\varepsilon=T^2\times\{\varepsilon\}\subset \mathrm{int}N_*$ with dividing set consisting two curves of slope ∞. Let L be a Legendrian ruling curve on T_ε with slope -1 (*c.f.* Remark 5.2.6). The Legendrian curve L has twisting number -1 with respect to the framing coming from T_ε. Let $(N',\Gamma',(\zeta_1^+)')$ be obtained from (N_*,Γ_*,ζ_1^+) by a contact (+1)-surgery along L. By [BS16b, Theorem 4.6], the cobordism map Φ corresponding to the contact (+1)-surgery that sends $\theta(N_*,\Gamma_*,\zeta_1^+)$ to $\theta((N',\Gamma',(\zeta_1^+)'))=0$. By [GHVHM08, Lemma 7], the resulting contact structure $(\zeta_1^+)'$ is overtwisted. Hence by [BS16b, Theorem 1.3], we have $\theta((N',\Gamma',(\zeta_1^+)'))=0$. It remains to show Φ is injective (at least on the subspace generated by $\theta(N_*,\Gamma_*,\zeta_1^+)$.

Write (N_*,Γ_*,ζ_0^+) for N_0. In the proof of Lemma 5.2.8, by considering the relative spin^c structure, the authors of [GHVHM08] showed that $c(N_*,\Gamma_*,\zeta_0^+)$ and $c(N_*,\Gamma_*,\zeta_1^+)$ lie in the same $\mathbb{F}_2$ summand of $SFH(-N_*,-\Gamma_*)\cong\mathbb{F}_2^4$ (we replace $\mathbb{Z}$-summand by $\mathbb{F}_2$ summand for the naturality issue, *c.f.* Remark 5.2.3). The contact structure ζ_0^+ and the contact structure $(\zeta_0^+)'$ after the contact $(+1)$-surgery along L can be embedded into S^3 and $S^1\times S^2$ with standard tight contact structures, respectively, which are both Stein fillable. Then both $c(N_*,\Gamma_*,\zeta_0^+)$ and $c(N',\Gamma',(\zeta_0^+)')$ are non-vanishing. Thus, the map Φ is injective on the $\mathbb{F}_2$ summand generated by $c(N_*,\Gamma_*,\zeta_0^+)$.

For sutured instanton homology, the analog of the (nontorsion) relative spin^c decomposition is the decomposition associated to admissible surfaces, constructed in [GL19, Li19]. We can use two annuli

$$A_0=S^1\times\{\mathrm{pt}\}\times I, A_1=\{\mathrm{pt}\}\times S^1\times I\subset T^2\times I$$

to construct the decomposition, where the S^1 factors corresponding to curves of slopes ∞ and 0 parallel to the dividing sets, respectively. Since $|\partial A_i\cap\Gamma_*|=2$ for $i=0,1$, by Theorem 2.3.20, there are only two nontrivial gradings for A_i, corresponding to the sutured manifold decomposition along A_i and $-A_i$. It is straightforward to check that sutured manifold decomposition along $\pm A_0\cup\pm A_1$ gives a 3-ball with a connected suture, whose $\underline{\mathrm{SHI}}$ is 1-dimensional. Thus,

$$\dim_{\mathbb{C}}\underline{\mathrm{SHI}}(-N_*,-\Gamma_*)=4.$$

By Proposition 5.2.9, we know that $\theta(N_*,\Gamma_*,\zeta_1^+)$ and $\theta(N_*,\Gamma_*,\zeta_0^+)$ live in the same grading. Since $\underline{\mathrm{SHI}}$ is 1-dimensional in any nontrivial grading, the elements $\theta(N_*,\Gamma_*,\zeta_1^+)$ and $\theta(N_*,\Gamma_*,\zeta_0^+)$ are linear dependent. By [BS16b, Corollary 1.6] and the Stein fillablility, both $\theta(N_*,\Gamma_*,\zeta_0^+)$ and $\theta(N',\Gamma',(\zeta_0^+)')$ are non-vanishing. Then Φ is injective on the subspace generated by $\theta(N_*,\Gamma_*,\zeta_0^+)$, and $\Phi(\theta(N_*,\Gamma_*,\zeta_1^+))=0$ implies $\theta(N_*,\Gamma_*,\zeta_1^+)=0$.

□

Proof of Theorem 5.2.13. This follows from Lemma 5.2.5, Corollary 5.2.15, and Proposition 5.2.16. Note that Lemma 5.2.5 is only about contact topology, so we can apply it without change. □

5.2.4 Vanishing results about cobordism maps

Suppose $(M,\gamma) \subset (M',\gamma')$ is a proper inclusion of balanced sutured manifolds and suppose ξ is a contact structure compatible with $(M'\backslash \mathrm{int} M, \gamma' \cup (-\gamma))$. By Corollary 5.2.15, if

$$\theta(M'\backslash \mathrm{int} M, \gamma' \cup (-\gamma), \xi) = 0,$$

then the contact gluing map Φ_ξ vanishes on the subspace of $\underline{\mathrm{SHI}}(-M,-\gamma)$ generated by instanton contact elements. Indeed, we can prove a stronger result by the functoriality of Φ_ξ. The proof of the following proposition is due to Ian Zemke.

Proposition 5.2.17. *Suppose $(M,\gamma) \subset (M',\gamma')$ is a proper inclusion of balanced sutured manifolds and suppose ξ is a contact structure compatible with*

$$(M_0,\gamma_0) := (M'\backslash \mathrm{int} M, \gamma' \cup (-\gamma)).$$

If the contact element $\theta(M_0,\gamma_0,\xi)$ vanishes, then the map Φ_ξ vanishes on $\underline{\mathrm{SHI}}(-M,-\gamma)$.

Proof. We have inclusions

$$(M,\gamma) \subset (M,\gamma) \sqcup (M_0,\gamma_0) \subset (M',\gamma'),$$

where $\sqcup$ denotes the disjoint union. The manifold

$$M'\backslash \mathrm{int}(M \sqcup M_0)$$

is contactomorphic to $\partial M \times I$. Let ξ_0 be the product contact structure on $\partial M \times I$. By the connected sum formula [Li20, Section 1.8], we have

$$\underline{\mathrm{SHI}}(-M \sqcup (-M_0), -\gamma \sqcup (-\gamma_0)) \cong \underline{\mathrm{SHI}}(-M,-\gamma) \otimes \underline{\mathrm{SHI}}(-M_0,-\gamma_0).$$

By functoriality, the map Φ_ξ is the composition of the following maps

$$\begin{array}{ccccc} \underline{\mathrm{SHI}}(-M,-\gamma) & \to & \underline{\mathrm{SHI}}(-M,-\gamma) \otimes \underline{\mathrm{SHI}}(-M_0,-\gamma_0) & \to & \underline{\mathrm{SHI}}(-M',-\gamma') \\ x & \mapsto & x \otimes \theta(M_0,\gamma_0,\xi) & \mapsto & \Phi_{\xi_0}(x \otimes \theta(M_0,\gamma_0,\xi)). \end{array}$$

If $\theta(M_0,\gamma_0,\xi) = 0$, then $\Phi_\xi = 0$. □

Remark 5.2.18. For a general balanced sutured manifold (M,γ), instanton contact elements do not generate $\underline{\mathrm{SHI}}(-M,-\gamma)$ because the number of tight contact structures compatible with (M,γ) is less than $\dim_{\mathbb{C}} \underline{\mathrm{SHI}}(M,\gamma)$. See [Li19, Section 4.3] and [Hon00] for discussion about contact structures on the solid torus.

The following vanishing result is used in the next section.

Corollary 5.2.19. *Suppose $(M,\gamma) \subset (M',\gamma')$ is a proper inclusion of balanced sutured manifolds. If*

$$(M'\backslash \mathrm{int} M, \gamma' \cup (-\gamma), \xi) = (N_*, \Gamma_*, \zeta_1^+) \text{ or } (N_*, \Gamma_*, \zeta_1^-)$$

defined in Subsection 5.2.1, then $\Phi_\xi = 0$.

Proof. This follows from Proposition 5.2.16 and Proposition 5.2.17 □

5.3 Instanton L-space knots

In this section, we study the instanton knot homology of an instanton L-space knot $K \subset Y$ and prove Theorem 1.3.3. For technical reasons, we only deal with the case $H_1(Y\backslash K) \cong \mathbb{Z}$.

5.3.1 The dimension in each grading

In this subsection, we prove the following theorem. The main input is the large surgery formula and the vanishing result Corollary 5.2.19.

Theorem 5.3.1. *Suppose Y is an integral homology sphere with $I^\sharp(Y) \cong \mathbb{C}$. Suppose $K \subset Y$ is a knot and S is the Seifert surface of K. If there is a positive integer n so that $Y_{-n}(K)$ is an instanton L-space, then for any $i \in \mathbb{Z}$, we have*

$$\dim_{\mathbb{C}} \underline{\mathrm{KHI}}(-Y, K, S, i) \leq 1.$$

Since Y is an integral homology sphere, K is always null-homologous and $\hat{\mu} = \mu, \hat{\lambda} = \lambda$ in Subsection 5.1.1. By Definition 5.1.2, we have $(q,p) = (1,0)$ and $(q_0,p_0) = (0,1)$. Then we have

$$\widehat{\Gamma}_\mu = \Gamma_\mu = \gamma_\mu, \quad \widehat{\Gamma}_n = \Gamma_n = \gamma_{\lambda - n\mu}.$$

Note that in the proof of Theorem 5.1.27, an auxiliary slope $\hat{\mu}' = n\hat{\mu} - \hat{\lambda}$ is used. Here we set $\hat{\mu}' = n\mu - \lambda$. Since n is not fixed, this slope is also not fixed.

For simplicity, we write $\gamma_{(x,y)}$ for $\gamma_{x\lambda + y\mu}$ in Definition 5.1.2. Also, we omit S in the notation $\underline{\mathrm{SHI}}(-Y\backslash K, \gamma, S, i)$ for any γ.

Then we make the following definition.

Definition 5.3.2. For any integers n and i with $|i| \le g(K)$, define

$$T_{n,i} = \underline{\mathrm{SHI}}(-Y\backslash K, -\Gamma_n, i + \frac{n-1}{2}),$$
$$B_{n,i} = \underline{\mathrm{SHI}}(-Y\backslash K, -\Gamma_n, i - 1 - \frac{n-1}{2}).$$

For $i > g(K)$ and any n, define $T_{n,i} = 0$. For $i < -g(K)$ and any n, define $B_{n,i} = 0$.

Remark 5.3.3. The notations 'T' and 'B' mean 'top' and 'bottom'. If we use the notations after the diagram (5.1.11) and suppose $g = g(K)$, then for any integers n and i with $|i| \le g(K)$, we have

$$T_{n,i} = \widehat{\Gamma}_n^{i,+} \text{ and } B_{n,i} = \widehat{\Gamma}_{n-1}^{i,-}.$$

By Lemma 5.1.12, we have

$$\psi^n_{-,n+1} : T_{n,i} \xrightarrow{\cong} T_{n+1,i} \text{ and } \psi_{+,n+1} : B_{n,i} \xrightarrow{\cong} B_{n+1,i}$$

for $n \ge 2g(K) + 1$ and $|i| \le g(K)$.

The following proposition follows from the large surgery formula.

Proposition 5.3.4. *Suppose Y is an integral homology sphere with $I^\sharp(Y) \cong \mathbb{C}$. Suppose $K \subset Y$ is a knot. Suppose n is an integer so that $n \ge 2g(K) + 1$ and $Y_{-n}(K)$ is an instanton L-space. Then we have the following.*

$$\underline{\mathrm{SHI}}(-Y\backslash K, -\gamma_{(2,1-2n)}, i) \cong \begin{cases} T_{n,i-n+1} & n - g \le i \le n - 1 + g \\ \mathbb{C} & -n + g + 1 \le i \le n - g - 1 \\ B_{n,i+n-1} & -n + 1 - g \le i \le -n + g \end{cases}$$

Proof. The isomorphism of the top and bottom $2g$ gradings of $\underline{\mathrm{SHI}}(-Y\backslash K, -\gamma_{(2,1-2n)})$ follows from applying Lemma 5.1.12 to $\hat{\mu}'$. Since $Y_{-n}(K)$ is an instanton L-space, by Term (5) of Theorem 2.3.20, the manifold $-Y_{-n}(K)$ is also an instanton L-space. The isomorphism of the middle gradings follows from Proposition 5.1.17, Lemma 5.1.13, and Theorem 1.3.10. □

Note that in the proof of Theorem 5.1.27 (more precisely, in the triangle (5.1.18)), we have a map $\psi^{\mu}_{-,0}(\hat{\mu}')$ from the space associated to $\widehat{\Gamma}_n$ to the space associated to $\widehat{\Gamma}_{n-1}$. We write this map as $\psi^n_{-,n-1}$. We also write $\psi^{2n-1}_{-,n}$ and $\psi^{n-1}_{-,2n-1}$ for $\psi^1_{-,\mu}(\hat{\mu}')$ and $\psi^0_{-,1}(\hat{\mu}')$ in (5.1.18), respectively. Similarly we write $\psi^n_{+,n-1}, \psi^{2n-1}_{+,n}$, and $\psi^{n-1}_{+,2n-1}$ for maps in the positive bypass triangle. We abuse notation so that bypass maps also denote their restrictions on a single grading. Then the following proposition follows from the vanishing results established in Section 5.2.

Proposition 5.3.5. *Suppose $K \subset Y$ is a null homologous knot. For any integer $n \in \mathbb{Z}$ with $n \geq 2g(K)+1$ and any integer i with $|i| \leq g(K)$, we have*

$$\psi^{n+1}_{+,n} \circ \psi^{n+2}_{-,n+1} = 0 : T_{n+2,i} \to T_{n,i}$$

and

$$\psi^{n+1}_{-,n} \circ \psi^{n+2}_{+,n+1} = 0 : B_{n+2,i} \to B_{n,i}.$$

Proof. By Remark 5.3.3, it suffices to prove

$$\Psi_T := \psi^{n+2}_{-,n+3} \circ \psi^{n+1}_{-,n+2} \circ \psi^{n}_{-,n+1} \circ \psi^{n+1}_{+,n} \circ \psi^{n+2}_{-,n+1} = 0 : T_{n+2,i} \to T_{n+3,i}$$

and

$$\Psi_B := \psi^{n+2}_{+,n+3} \circ \psi^{n+1}_{+,n+2} \circ \psi^{n}_{+,n+1} \circ \psi^{n+1}_{-,n} \circ \psi^{n+2}_{+,n+1} = 0 : B_{n+2,i} \to B_{n+3,i}.$$

By classification of tight contact structures on $T^2 \times I$ [Hon00], we know that the contact structures corresponding to Ψ_T and Ψ_B are contactomorphic to either $(N_*, \Gamma_*, \zeta_1^+)$ or $(N_*, \Gamma_*, \zeta_1^-)$ defined in Subsection 5.2.1. Then the lemma follows from Corollary 5.2.19. □

Proposition 5.3.6. *Suppose Y is an integral homology sphere with $I^\sharp(Y) \cong \mathbb{C}$. Suppose $K \subset Y$ is a knot. Suppose n_0 be a positive integer so that $Y_{-n_0}(K)$ is an instanton L-space. Then for any integer n so that $n > n_0$, $Y_{-n}(K)$ is also an instanton L-space.*

Proof. This proposition follows immediately from $\chi(I^\sharp(Y_{-n}(K)) = |H_1(Y_{-n}(K))|$, the equation

$$|H_1(Y_{-n-1}(K))| = |H_1(Y_{-n}(K))| + |H_1(Y)|,$$

and the following surgery exact triangle ([BS18, Section 4.2], see also [Sca15])

$$\begin{array}{ccc} I^\sharp(Y_{-n-1}(K)) & \longrightarrow & I^\sharp(Y_{-n}(K)) \\ & \nwarrow \quad \swarrow & \\ & I^\sharp(Y) & \end{array}$$

□

By Proposition 5.3.4 and Proposition 5.3.5, the proof of Theorem 5.3.1 follows from similar algebraic lemmas in [OS05b, Section 3]. We reprove them in our setting.

Lemma 5.3.7. *Suppose Y is an integral homology sphere with $I^{\sharp}(Y) \cong \mathbb{C}$. Suppose $K \subset Y$ is a knot. Suppose n_0 be a positive integer so that $Y_{-n_0}(K)$ is an instanton L-space. Suppose further that for a large enough integer n and some integer m with $|m| \leq g(K)$, we have $T_{n,m+1} = 0$. Then one of the following two cases happens.*

(1) $\underline{\mathrm{KHI}}(-Y,K,m) \cong \mathbb{C}$ *and* $B_{n,m-1} = 0$,

(2) $\underline{\mathrm{KHI}}(-Y,K,m) = 0$ *and* $T_{n,m} = 0$.

Proof. By Proposition 5.3.6, we can take an arbitrary large enough integer n, since they are all L-space surgery slopes. From Proposition 5.1.10, we have the following exact triangle

$$\begin{array}{ccc} T_{n-1,m+1} & \longrightarrow & T_{n,m} \\ & \nwarrow \quad \swarrow & \\ & \underline{\mathrm{KHI}}(-Y,K,m) & \end{array}$$

From Remark 5.3.3 and the assumption $T_{n,m+1} = 0$, we know that

$$T_{n-1,m+1} \cong T_{n,m+1} = 0 \text{ and } B_{n,m-1} \cong B_{n-1,m-1}.$$

Hence there exists some $k \in \mathbb{N}$ so that

$$T_{n,m} \cong \underline{\mathrm{KHI}}(-Y,K,m) \cong \mathbb{C}^k.$$

Also from Proposition 5.1.10, we have the following exact diagram

$$\begin{array}{ccccc} & & \underline{\mathrm{SHI}}(-Y\backslash K, -\gamma_{(2,1-2n)}, m) & & \\ & & \downarrow & & \\ & & T_{n,m} & & \\ & & \downarrow{\scriptstyle \psi^{n,m}_{-,n-1}} & & \\ \underline{\mathrm{SHI}}(-Y\backslash K, -\gamma_{(2,3-2n)}, m-1) & \xrightarrow{\psi^{2n-3,m-1}_{+,n-1}} & B_{n-1,m-1} & \xrightarrow{\psi^{n-1,m-1}_{+,n-2}} & T_{n-2,m} \end{array}$$

where $\psi^{n,m}_{-,n-1}$ is the map $\psi^{n}_{-,n-1}$ restricted to the graded part $T_{n,m}$ and other notations are defined similarly. Since $|m| \leq g(K)$, Proposition 5.3.4 implies that

$$\underline{\mathrm{SHI}}(-Y\backslash K, -\gamma_{(2,1-2n)}, m) \cong \underline{\mathrm{SHI}}(-Y\backslash K, -\gamma_{(2,3-2n)}, m-1) \cong \mathbb{C}.$$

Hence the above diagram can be rewritten as

$$\begin{array}{ccccc} & & \mathbb{C} & & \\ & & \big\downarrow & & \\ & & T_{n,m} \cong \mathbb{C}^k & & \\ & & \big\downarrow{\scriptstyle \psi^{n,m}_{-,n-1}} & & \\ \mathbb{C} & \xrightarrow{\psi^{2n-3,m-1}_{+,n-1}} & B_{n-1,m-1} & \xrightarrow{\psi^{n-1,m-1}_{+,n-2}} & T_{n-2,m} \cong \mathbb{C}^k \end{array} \tag{5.3.1}$$

We consider the following two cases.

Case 1. $\psi^{2n-3,m-1}_{+,n-1}$ is trivial. Then from the exactness of the horizontal sequence in (5.3.1), we know that $B_{n-1,m-1} \cong \mathbb{C}^{k-1}$ and $\psi^{n-1,m-1}_{+,n-2}$ is injective. Also, we conclude from the exactness of the vertical sequence in (5.3.1) that $\psi^{n,m}_{-,n-1}$ is surjective. However, from Proposition 5.3.5 we know that

$$\psi^{n-1,m-1}_{+,n-2} \circ \psi^{n,m}_{-,n-1} = 0.$$

Hence the only possibility is that $k = 1$, and this concludes that $T_{n,m} \cong \underline{\mathrm{KHI}}(-Y, K, m) \cong \mathbb{C}$, and $B_{n,m-1} \cong B_{n-1,m-1} = 0$, which is the first case in the statement of the lemma.

Case 2. $\psi^{2n-3,m-1}_{+,n-1}$ is nontrivial. Then from the exactness of the horizontal sequence in (5.3.1), we know that $B_{n-1,m-1} \cong \mathbb{C}^{k+1}$ and $\psi^{n-1,m-1}_{+,n-2}$ is surjective. From the above discussion and the bypass exact triangle from Proposition 5.1.10, we have another exact diagram

$$\begin{array}{ccc} & & \underline{\mathrm{SHI}}(-Y\backslash K, -\gamma_{(2,5-2n),m}) \cong \mathbb{C} \\ & & \big\downarrow \\ B_{n-1,m-1} \cong \mathbb{C}^{k+1} & \xrightarrow{\psi^{n-1,m-1}_{+,n-2}} & T_{n-2,m} \cong \mathbb{C}^k \\ & & \big\downarrow{\scriptstyle \psi^{n-2,m}_{-,n-3}} \\ & & B_{n-3,m-1} \cong \mathbb{C}^{k+1} \end{array} \tag{5.3.2}$$

The exactness of the vertical sequence in (5.3.2) implies that the map $\psi^{n-2,m}_{-,n-3}$ is injective. However, from Proposition 5.3.5, we have

$$\psi^{n-2,m}_{-,n-3} \circ \psi^{n-1,m-1}_{+,n-2} = 0.$$

Hence the only possibility is that $k = 0$. Thus, we conclude that $T_{n,m} \cong \underline{\mathrm{KHI}}(-Y, K, m) = 0$, which is the second case in in the statement of the lemma. □

Lemma 5.3.8. *Suppose Y is an integral homology sphere with $I^\sharp(Y) \cong \mathbb{C}$. Suppose $K \subset Y$ is a knot. Suppose n_0 be a positive integer so that $Y_{-n_0}(K)$ is an instanton L-space. Suppose further that for a large enough integer n and some integer m with $|m| \le g(K)$, we have $B_{n,m} = 0$. Then one of the following two cases happens.*

(1) $\underline{\mathrm{KHI}}(-Y, K, m) \cong \mathbb{C}$ *and* $T_{n,m} = 0$*,*

(2) $\underline{\mathrm{KHI}}(-Y, K, m) = 0$ *and* $B_{n,m-1} = 0$*.*

Proof. The proof is similar to that of Lemma 5.3.7. From Proposition 5.1.10, we have the following triangle

$$\begin{array}{ccc} B_{n-1,m-1} & \longrightarrow & B_{n,m} \\ & \nwarrow \quad \swarrow & \\ & \underline{\mathrm{KHI}}(-Y,K,m) & \end{array}$$

Hence there exists some $k \in \mathbb{N}$ so that

$$B_{n-1,m-1} \cong \underline{\mathrm{KHI}}(-Y, K, m) \cong \mathbb{C}^k.$$

Also from Proposition 5.1.10, we have the following exact diagram

$$\begin{array}{ccccc} & & \mathbb{C} & & \\ & & \downarrow & & \\ & & T_{n,m} & & \mathbb{C} \\ & & \downarrow{\scriptstyle \psi^{n,m}_{-,n-1}} & & \downarrow \\ \mathbb{C} & \xrightarrow{\psi^{2n-3,m-1}_{+,n-1}} & B_{n-1,m-1} \cong \mathbb{C}^k & \xrightarrow{\psi^{n-1,m-1}_{+,n-2}} & T_{n-2,m} \\ & & & & \downarrow{\scriptstyle \psi^{n-2,m-1}_{-,n-3}} \\ & & & & B_{n-3,m-1} \cong \mathbb{C}^k \end{array} \tag{5.3.3}$$

We consider the following two cases.

Case 1. $\psi^{2n-3,m-1}_{+,n-1}$ is trivial. Then from the exactness of the horizontal sequence in (5.3.3), we know that $T_{n-2,m} \cong \mathbb{C}^{k-1}$ and $\psi^{n-1,m-1}_{+,n-2}$ is surjective. Also, we conclude from the exactness of the second vertical sequence in (5.3.3) that $\psi^{n-2,m}_{-,n-3}$ is injective. However, from Proposition 5.3.5 we know that

$$\psi^{n-2,m}_{-,n-3} \circ \psi^{n-1,m-1}_{+,n-2} = 0.$$

Hence the only possibility is that $k = 1$. Hence we conclude that $\underline{\mathrm{KHI}}(-Y, K, m) \cong \mathbb{C}$ and $T_{n,m} \cong T_{n-2,m} = 0$, which is the first case in the statement of the lemma.

Case 2. $\psi^{2n-3,m-1}_{+,n-1}$ is nontrivial. Then from the exactness of the horizontal sequence in (5.3.3), we know that $T_{n,m}\cong T_{n-2,m}\cong\mathbb{C}^{k+1}$ and $\psi^{n-1,m-1}_{+,n-2}$ is injective. Also, we conclude from the exactness of the first vertical sequence that $\psi^{n,m}_{-,n-1}$ is surjective. However, from Proposition 5.3.5 we know that

$$\psi^{n-1,m-1}_{+,n-2}\circ\psi^{n,m}_{-,n-1}=0.$$

Hence the only possibility is that $k=0$, and this concludes that

$$B_{n,m-1}\cong B_{n-1,m-1}\cong\underline{\mathrm{KHI}}(-Y,K,m)\cong\mathbb{C}^k,$$

which is the second case in the statement of the lemma. □

Proof of Theorem 5.3.1. By Definition 5.3.2 and Lemma 5.1.8, we know that

$$T_{n,g(K)+1}=0 \text{ and } \underline{\mathrm{KHI}}(-Y,K,g(K)+1)=0.$$

We apply an induction that decreases the integer i: assuming that for $i+1$, we have

$$\underline{\mathrm{KHI}}(-Y,K,i+1)\cong\mathbb{C} \text{ or } 0$$

and either $T_{n,i+1}=0$ or $B_{n,(i+1)-1}=0$, then we want to prove the same results for i. When $T_{n,i+1}=0$, from Lemma 5.3.7, we have either $\underline{\mathrm{KHI}}(-Y,K,i)\cong\mathbb{C}$ and $B_{n,i-1}=0$ or $\underline{\mathrm{KHI}}(-Y,K,i)=0$ and $T_{n,i}=0$. When $B_{n,(i+1)-1}=0$, from Lemma 5.3.8, we have either $\underline{\mathrm{KHI}}(-Y,K,i)\cong\mathbb{C}$ and $T_{n,i}=0$ or $\underline{\mathrm{KHI}}(-Y,K,i)=0$ and $B_{n,i-1}=0$. Hence, the inductive step is completed and we conclude that

$$\underline{\mathrm{KHI}}(-Y,K,i)\cong\mathbb{C} \text{ or } 0.$$

for all $i\in\mathbb{Z}$ so that $|i|\le g(K)$. From Lemma 5.1.8, we know that

$$\underline{\mathrm{KHI}}(-Y,K,i)\cong 0$$

for all $i\in\mathbb{Z}$ with $|i|>g(K)$. Hence we conclude the proof of Theorem 5.3.1. □

5.3.2 Coherent chains

In this subsection, we prove instanton analog of [RR17, Lemma 3.2] with more assumptions. First, we introduce the analog of [RR17, Definition 3.1] in instanton theory.

Definition 5.3.9. Suppose K is a knot in a rational homology sphere Y and suppose $\hat{\mu}$ is the meridian of K. Suppose the knot complement $Y\backslash K$ satisfying $H_1(Y\backslash K)\cong\mathbb{Z}$ so that we can identify $[\hat{\mu}]\in H_1(Y\backslash K)$ as an integer q. Indeed, if a Seifert surface S of K is chosen, we can set $q=S\cdot\hat{\mu}$. For any integer s and its image $[s]\in\mathbb{Z}_q$, define

$$\underline{\mathrm{KHI}}(-Y,K,[s]):=\bigoplus_{k\in\mathbb{Z}}\underline{\mathrm{KHI}}(-Y,K,S,s+kq).$$

It is called a **positive chain** if it is generated by elements

$$x_1,\dots,x_l,y_1,\dots,y_{l-1},$$

each of which lives in a single grading associated to S and a single $\mathbb{Z}_2$-grading, and the differentials d_+ and d_- satisfy

$$d_-(y_i)\doteq x_{i+1},\quad d_+(y_i)\doteq x_i,\text{ and }d_-(x_i)=d_+(x_i)=0\text{ for all }i,$$

where $\doteq$ means equal up to multiplication by a unit. The space $\underline{\mathrm{KHI}}(-Y,K,[s])$ is called a **negative chain** if there exist similar generators so that

$$d_-(x_i)\doteq y_i,\quad d_+(x_i)\doteq y_{i-1},\text{ and }d_-(y_i)=d_+(y_i)=0\text{ for all }i.$$

We call $\underline{\mathrm{KHI}}(-Y,K)$ **consists of positive chains** if $\underline{\mathrm{KHI}}(-Y,K,[s])$ is a positive chain for any $[s]\in\mathbb{Z}_q$ and **consists of negative chains** if $\underline{\mathrm{KHI}}(-Y,K,[s])$ is a negative chain for any $[s]\in\mathbb{Z}_q$. We call $\underline{\mathrm{KHI}}(-Y,K)$ **consists of coherent chains** if $\underline{\mathrm{KHI}}(-Y,K)$ either consists of postive chains or consists of negative chains

Remark 5.3.10. By Definition 5.3.9, the space $\underline{\mathrm{KHI}}(-Y,K,[s])$ is both a positive chain and a negative chain if and only if $\dim_{\mathbb{C}}\underline{\mathrm{KHI}}(-Y,K,[s])=1$. By the proof of Proposition 5.1.37, the space $\underline{\mathrm{KHI}}(-Y,K)$ consists of positive chains if and only if $\underline{\mathrm{KHI}}(Y,K)$ consists of negative chains.

The main theorem in this subsection is the following.

Theorem 5.3.11. *Suppose $K\subset Y$ is a knot as in Definition 5.3.9. Note that $H_1(Y\backslash K)\cong\mathbb{Z}$. Suppose Y is an instanton L-space and suppose $n\in\mathbb{N}_+$. Suppose the basis $(\hat{\mu},\hat{\lambda})$ of $\partial Y\backslash K$ is from Definition 5.1.2. If $Y_{-n}(K)$ is an instanton L-space, then $\underline{\mathrm{KHI}}(-Y,K)$ consists of positive chains. If $Y_n(K)$ is an instanton L-space, then $\underline{\mathrm{KHI}}(-Y,K)$ consists of negative chains.*

For simplicity, we only provide details of the proof for a special case of Theorem 5.3.11. The proof for the general case is similar. The main input is Theorem 5.3.1.

Definition 5.3.12. We adapt notations in Subsection 5.3.1 and Construction 5.1.25. For any integer s, suppose $B^+_{\geq s}$ is the subcomplex of B^+_s with the underlying space

$$\bigoplus_{k\geq s}\underline{\mathrm{SHI}}(-Y\backslash K,-\widehat{\Gamma}_\mu,S,s+kq)$$

and suppose $B^-_{<s}$ is the subcomplex of B^-_s with the underlying space

$$\bigoplus_{k<s}\underline{\mathrm{SHI}}(-Y\backslash K,-\widehat{\Gamma}_\mu,S,s+kq).$$

Let $H(B^+_{\geq s})$ and $H(B^-_{<s})$ be the corresponding homologies.

Lemma 5.3.13. *For any integers n and i with $|i|\leq g(K)$, we have*

$$T_{n,i}\cong H(B^+_{\geq i}) \text{ and } B_{n,i}\cong H(B^-_{<i}).$$

Proof. This follows from Remark 5.3.3, equations (5.1.12) and (5.1.13), and Theorem 2.2.6. □

Theorem 5.3.14. *Suppose K is a knot in an integral homology sphere Y with* $\dim_{\mathbb{C}} I^\sharp(Y)=1$. *If there is a positive integer n so that $Y_{-n}(K)$ is an instanton L-space, then* $\underline{\mathrm{KHI}}(-Y,K)$ *consists of positive chains in the sense of Definition 5.3.9.*

Proof. By Theorem 5.3.1, for any integer i, we have

$$\dim_{\mathbb{C}}\underline{\mathrm{KHI}}(-Y,K,i)\leq 1.$$

Then we have integers

$$n_1>n_2>\cdots>n_k$$

so that

$$\dim_{\mathbb{C}}\underline{\mathrm{KHI}}(-Y,K,i)=\begin{cases}1 & \text{if } i=n_j \text{ for } j\in[0,k];\\ 0 & \text{else.}\end{cases}$$

Suppose x_i is the generator of $\underline{\mathrm{KHI}}(-Y,K,n_{2i-1})$ and y_i is the generator of $\underline{\mathrm{KHI}}(-Y,K,n_{2i})$. We verify that those x_i and y_i satisfy the positive chain condition, *i.e.* for any integer i, we have

$$d_-(y_i)\doteq x_{i+1}, d_+(y_i)\doteq x_i, \text{ and } d_-(x_i)=d_+(x_i)=0, \tag{5.3.4}$$

where $\doteq$ means the equation holds up to multiplication by a unit. We prove this condition by induction. We only consider the condition about the differential d_+. The proof for d_- is similar. The gradings in the following arguments mean the gradings associated to the Seifert surface S. Note that by the proof of Theorem 5.3.1, we have

$$T_{n,n_{2l}} = B_{n,n_{2l-1}+1} = 0 \text{ for any } l.$$

Hence by Lemma 5.3.13, we have

$$T_{n,i} \cong H(B^+_{\geq n_{2l}}) = H(B^-_{<2l-1}).$$

First, suppose $i = 1$. Since x_1 lives in the top grading of $\underline{\mathrm{KHI}}(-Y,K)$ and d_+ increases the $\mathbb{Z}$-grading, we must have $d_+(x_1) = 0$. Since $H(B^+_{\geq n_2}) = 0$ and there are only two generators x_1 and y_1 in $B^+_{\geq n_2}$, we must have $d_+(y_1) \doteq x_1$.

Then we assume the condition (5.3.4) holds for $i \leq l-1$ and prove it also holds for $i = l$. Since

$$H(B^+_{\geq n_{2l}}) = H(B^+_{\geq n_{2l-2}}) = 0,$$

we know the quotient complex $B^+_{\geq n_{2l}}/B^+_{\geq n_{2l-2}}$ also has trivial homology. Since it is generated by x_l and y_l, the coefficient of $d_+(y_l)$ about x_l must be nontrivial. Hence y_l is not in the $(n_{2l-1} - n_{2l} + 1)$-page of the spectral squence associated to d_+. Since other generators $x_1, \ldots, x_{l-1}, y_1, \ldots, y_{l-1}$ have smaller gradings than x_l, we know by construction of d_+ in Construction 2.2.7 that the coefficients of $d_+(y_l)$ about those generators are zeros. Hence $d_+(y_l) \doteq x_l$. Since $d_+ \circ d_+ = 0$, we have $d_+(x_l) = 0$. Thus, we prove the condition holds for $i = l$.

□

Proof of Theorem 5.3.11. If $Y_{-n}(K)$ is an instanton L-space, then the proof is similar to that of Theorem 5.3.14. To prove a generalization of Theorem 5.3.1, we need to remove the integral homology sphere assumption in Proposition 5.1.17 and Proposition 5.3.6. The corresponding proofs follow from the proofs of Proposition 5.1.17 and [BGW13, Proposition 4]. If $Y_n(K)$ is an instanton L-space, by Remark 5.3.10, we can consider the mirror knot to obtain the result. □

5.3.3 A graded version of Künneth formula

In this subsection, we prove the following graded version of Künneth formula for the connected sum of two knots.

Proposition 5.3.15. *Suppose Y_1 and Y_2 are two irreducible rational homology spheres and $K_1 \subset Y_1$, $K_2 \subset Y_2$ are two knots so that $Y_1 \backslash K_1$ and $Y_2 \backslash K_2$ are both irreducible. Suppose*

$$(Y', K') = (Y_1 \sharp Y_2, K_1 \sharp K_2)$$

is the connected sum of two knots. Then there is a minimal genus Seifert surface S of K' with the following properties.

(1) There is a 2-sphere $\Sigma \subset Y'$ intersecting the knot K' in two points and intersecting S in arcs.

(2) If we cut S along $S \cap \Sigma^2$, then S decomposes into two surfaces $S_1 \subset Y_1$ and $S_2 \subset Y_2$ so that S_i is a union of some copies of Seifert surfaces of K_i for $i = 1, 2$.

(3) There is an isomorphism

$$\underline{\mathrm{KHI}}(Y', K', S, k) \cong \bigoplus_{i+j=k} \underline{\mathrm{KHI}}(Y_1, K_1, S_1, i) \otimes \underline{\mathrm{KHI}}(Y_2, K_2, S_2, j). \tag{5.3.5}$$

Proof. Let S be a minimal genus Seifert surface of K' and let $\Sigma \subset Y'$ be a 2-sphere so that Σ intersects K' in two points. We can choose Σ so that

$$\Sigma \cap \partial Y' \backslash K' = \mu_1 \cup \mu_2,$$

where μ_1 and μ_2 are two meridians of K'. Write

$$A = \Sigma \cap Y' \backslash K'.$$

From now on, we also regard S as a surface inside the knot complement $Y' \backslash K'$. We can isotope S so that S intersects A transversely and S has minimal intersections with both μ_1 and μ_2. Now we argue that we can further isotope S so that S intersects A in arcs. Suppose

$$S \cap A = \alpha_1 \cup \cdots \cup \alpha_n \cup \beta_1 \cup \cdots \cup \beta_m,$$

where α_i are arcs and β_j are closed curves. Observe that each component of $A \backslash (\alpha_1 \cup \cdots \cup \alpha_n)$ is a disk. Then using the arguments in the proof of [Rol90, Chapter 5, Theorem A14], we could further assume that $m = 0$, *i.e.*, S intersects A in arcs. When we cut the knot complement $Y' \backslash K'$ along A, we obtain the disjoint union of the knot complements $Y_1 \backslash K_1$ and $Y_2 \backslash K_2$, and the surface S decomposes into $S_1 \subset Y_1 \backslash K_1$ and $S_2 \subset Y_2$. Note that S_1 and S_2 must be the union of (possibly more than one) copies of Seifert surfaces of the corresponding knots. Then we prove the isomorphism (5.3.5).

First, we prove

$$\underline{\mathrm{KHI}}(Y',K') \cong \underline{\mathrm{KHI}}(Y_1,K_1)\otimes\underline{\mathrm{KHI}}(Y_2,K_2). \tag{5.3.6}$$

To do so, we pick a meridian μ_i' of K_i for $i=1,2$ pick suitable orientations so that $(Y'\backslash K',\mu_1'\cup\mu_2')$ is a balanced sutured manifold. Then we can decompose it along the annulus A:

$$(Y'\backslash K',\mu_1'\cup\mu_2')\rightsquigarrow(Y_1\backslash K_1,\mu_1\cup\mu_1')\sqcup(Y_2\backslash K_2,\mu_2\cup\mu_2').$$

From [KM10b, Proposition 6.7], this annular decomposition leads to the isomorphism (5.3.6). To study the grading behavior of this isomorphism, we sketch the construction of the isomorphism as follows. Pick a connected oriented compact surface T so that

$$\partial T=-\mu_1\cup-\mu_2.$$

Pick an annulus T' so that

$$\partial T'=-\mu_1'\cup-\mu_2'.$$

One could think of T' be a copy of the annulus A.

In [KM10b, Section 7], Kronheimer and Mrowka constructed closures of

$$(Y_1\backslash K_1,\mu_1\cup\mu_1')\sqcup(Y_2\backslash K_2,\mu_2\cup\mu_2')$$

as follows. First, glue $[-1,1]\times(T\cup T')$ to $Y_1\backslash K_1\sqcup Y_2\backslash K_2$ using the boundary identifications as above to obtain a pre-closure

$$\widetilde{M}=(Y_1\backslash K_1\sqcup Y_2\backslash K_2)\cup[-1,1]\times(T\cup T'). \tag{5.3.7}$$

The boundary of $\widetilde{M}$ has two components

$$\partial\widetilde{M}=R_+\cup R_-,$$

where

$$R_\pm=R_\pm(\mu_1\cup\mu_1')\cup R_\pm(\mu_2\cup\mu_2')\cup\{\pm1\}\times(T\cup T').$$

Second, choose an orientation preserving diffeomorphism

$$h:R_+\to R_-$$

and use h to close up $\widetilde{M}$ and obtain a closed 3-manifold Y with a distinguishing surface R. The pair (Y,R) is a closure of $(Y_1\backslash K_1, \mu_1\cup\mu_1')\sqcup(Y_2\backslash K_2,\mu_2\cup\mu_2')$.

Remark 5.3.16. In [KM10b, Section 7], we also need to choose a simple closed curve in Y, either transversely intersecting R at one point or is non-separating on R, to achieve the irreducibility condition for related instanton moduli spaces. In the current proof, the choices of simple closed curves are straightforward, so we omit them from the discussion.

Note that gluing $[-1,1]\times T_1$ to $(Y_1\backslash K_1,\mu_1\cup\mu_1')\sqcup(Y_2\backslash K_2,\mu_2\cup\mu_2')$ is the inverse operation of decomposing $(Y'\backslash K',\mu_1'\cup\mu_2')$ along the annulus A. As a result, (Y,R) is clearly a closure of $(Y'\backslash K',\mu_1'\cup\mu_2')$ as well. The identification of the closures induces the isomorphism in (5.3.6). More precisely, we can pick the surface T with large enough genus and pick a simple closed curve $\theta\subset T$ so that θ separates T into two parts, both of large enough genus, and with $-\mu_1'$ and $-\mu_2'$ sitting in different parts. We also pick a core θ' of the annulus T'. When choosing the gluing diffeomorphism $h: R_+\to R_-$, we can choose one so that

$$h(\{1\}\times\theta)=\{-1\}\times\theta,\ \text{and}\ h(\{1\}\times\theta')=\{-1\}\times\theta'. \tag{5.3.8}$$

Hence, inside Y, there are two tori $S^1\times\theta$ and $S^1\times\theta'$. If we cut Y open along these two tori and reglue, then we obtain two connected 3-manifolds (Y_1,R_1) and (Y_2,R_2), which are closures of $(Y_1\backslash K_1,\mu_1\cup\mu_1')$ and $(Y_2\backslash K_2,\mu_2\cup\mu_2')$, respectively. The Floer's excision theorem in [KM10b, Section 7.3] then provide the desired isomorphism.

To study the gradings, recall that

$$S\cap A=\alpha_1\cup\cdots\cup\alpha_n$$

where α_i are arcs connecting μ_1 to μ_2 on A. We can also regard those arcs as on the annulus T'. Assume that ∂S intersects each of μ_1' and μ_2' in n points as well. Note that we have assumed that T has a large enough genus. Then there are arcs $\delta_1,\ldots,\delta_n$ so that the following holds. Recall we have chosen $\theta\subset T$ in previous above discussions.

(1) We have $\partial(\delta_1\cup\cdots\cup\delta_n)=S\cap(\mu_1'\cup\mu_2')$.

(2) For $i=1,..,n$, the arc δ_i intersects θ_1 transversely once.

(3) The surface $S\backslash(\delta_1\cup\cdots\cup\delta_n\cup\theta_1)$ also has two components.

(4) Let $\widetilde{S}=S\cup[-1,1]\times(\alpha_1\cup\cdots\cup\alpha_n)$ be a properly embedded surface inside the pre-closure $\widetilde{M}$ as in (5.3.7), then we can choose a gluing diffeomorphism $h: R_+\to R_-$ satisfying the

condition (5.3.8) and the following extra condition

$$h(\partial\widetilde{S}\cap R_{+})=\partial\widetilde{S}\cap R_{-}.$$

Hence, the surface S extends to a closed surface $\bar{S}\subset Y$ that induces the desired $\mathbb{Z}$-grading on $\underline{\mathrm{KHI}}(Y',K')$. When we cut Y open along $S^1\times\theta$ and $S^1\times\theta'$ and reglue, the surface $\bar{S}$ is also cut and reglued to form two closed surfaces $\bar{S}_1\subset Y_1$ and $\bar{S}_2\subset Y_2$. They are the extensions of the Seifert surface S_1 of K_1 and the Seifert surface S_2 of K_2 in the corresponding closures. Hence the Floer's excision theorem in [KM10b, Section 7.3] provides desired the isomorphism (5.3.5). □

Finally, we prove Theorem 1.3.3.

Proof of Theorem 1.3.3. By discussion in Section 1.3, we may assume $S^3_n(K)$ is an instanton L-space for some $n\in\mathbb{N}_+$. Then by Theorem 5.3.11, the space $\underline{\mathrm{KHI}}(S^3,K)$ consists of coherent chains. Then arguments about $KHI(S^3,K,S,i)$ follow from Definition 5.3.9 and Proposition 5.1.41.

To prove K is a prime knot, we can apply the proof of [BVV18, Corollary 1.4] to KHI, replacing [BVV18, Theorem 1.1] by [BS22, Theorem 1.7]. Note that we need the graded version of Künneth formula for KHI in Proposition 5.3.15. □

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Appendix A

Heegaard Floer theory

In this appendix, we collect constructions and properties of Heegaard Floer theory that are used in the main body of this dissertation. Most results are restatements of other people's work, while some involve direct calculations which can not be found elsewhere. The first section is about closed 3-manifolds and 4-dimensional cobordisms. The second section is about balanced sutured manifolds.

A.1 Heegaard Floer homology and the graph TQFT

A.1.1 Heegaard Floer homology for multi-pointed 3-manifolds

In this subsection and the next subsection, we provide an overview of the graph TQFT for Heegaard Floer theory, constructed by Zemke [Zem19] (see also [HMZ18, Zem20]), and list some properties which are relevant to proofs in the third subsection about Floer's excision theorem.

Definition A.1.1. A **multi-pointed 3-manifold** is a pair $(Y, \mathbf{w})$ consisting of a closed, oriented 3-manifold Y (not necessarily connected), together with a finite collection of basepoints $\boldsymbol{w} \subset Y$, such that each component of Y contains at least one basepoint.

Given two multi-pointed 3-manifolds $(Y_1, \boldsymbol{w}_1)$ and $(Y_2, \boldsymbol{w}_2)$, a **ribbon graph cobordism** from $(Y_1, \boldsymbol{w}_1)$ to $(Y_2, \boldsymbol{w}_2)$ is a pair (W, Γ) satisfying the following conditions.

(1) W is a cobordism from Y_1 to Y_2.

(2) Γ is an embedded graph in W such that $\Gamma \cap Y_i = \boldsymbol{w}_i$ for $i = 1, 2$. Furthermore, each point of $\boldsymbol{w}_i$ has valence 1 in Γ.

(3) Γ has finitely many edges and vertices, and no vertices of valence 0.

(4) The embedding of Γ is smooth on each edge.

(5) Γ is decorated with a formal ribbon structure, *i.e.*, a formal choice of cyclic ordering of the edges adjacent to each vertex.

Definition A.1.2. A ribbon graph cobordism (W,Γ) from $(Y_1,\boldsymbol{w}_1)$ to $(Y_2,\boldsymbol{w}_2)$ is called a **restricted graph cobordism** if W is obtained from $Y_1\times I$ by attaching 4-dimensional 1-, 2-, and 3-handles away from all basepoints and $\Gamma=\boldsymbol{w}_1\times I$ is the induced graph in W (so the cyclic ordering is unique and $|\boldsymbol{w}_1|=|\boldsymbol{w}_2|$).

Definition A.1.3 ([Zem19, Definition 4.1])**.** Suppose $(Y,\boldsymbol{w})$ is a connected multi-pointed 3-manifold. A **multi-pointed Heegaard diagram** $\mathcal{H}=(\Sigma,\alpha,\beta,\boldsymbol{w})$ for $(Y,\boldsymbol{w})$ is a tuple satisfying the following conditions.

(1) Σ is a closed, oriented surface, embedded in Y, such that $\boldsymbol{w}\subset\Sigma\backslash(\alpha\cup\beta)$. Furthermore, Σ splits Y into two handlebodies U_α and U_β, oriented so that $\Sigma=\partial U_\alpha=-U_\beta$.

(2) $\alpha=\{\alpha_1,\dots,\alpha_n\}$ is a collection of $n=g(\Sigma)+|\boldsymbol{w}|-1$ pairwise disjoint simple closed curves on Σ, bounding pairwise disjoint compressing disks in U_α. Each component of $\Sigma\backslash\alpha$ is planar and contains a single basepoint.

(3) $\beta=\{\beta_1,\dots,\beta_n\}$ is a collection of pairwise disjoint, simple, closed curves on Σ bounding pairwise disjoint compressing disks in U_β. Each component of $\Sigma\backslash\beta$ is planar and contains a single basepoint.

Suppose $\boldsymbol{w}=\{w_1,\dots,w_m\}$. Let the polynomial ring associated to $\boldsymbol{w}$ be

$$\mathbb{F}_2[U_{\boldsymbol{w}}]:=\mathbb{F}_2[U_{w_1},\dots,U_{w_m}].$$

Let $\mathbb{F}_2[U_{\boldsymbol{w}},U_{\boldsymbol{w}}^{-1}]$ be the ring obtained by formally inverting each of the variables.

If $\boldsymbol{k}=(k_1,\dots,k_m)$ is an m-tuple, let

$$U_{\boldsymbol{w}}^{\boldsymbol{k}}:=U_{w_1}^{k_1}\cdots U_{w_m}^{k_m}.$$

For simplicity, we will also write U_i for U_{w_i}.

Suppose $\mathcal{H}=(\Sigma,\alpha,\beta,\boldsymbol{w})$ is a multi-pointed Heegaard diagram of a connected multi-pointed 3-manifold $(Y,\boldsymbol{w})$. Suppose $n=g(\Sigma)+|\boldsymbol{w}|-1$. Consider two tori

$$\mathbb{T}_\alpha:=\alpha_1\times\cdots\times\alpha_n \text{ and } \mathbb{T}_\beta:=\beta_1\times\cdots\times\beta_n$$

in the symmetric product

$$\mathrm{Sym}^n\Sigma := (\prod_{i=1}^{n}\Sigma)/S_n.$$

The chain complex $CF^-(\mathcal{H})$ is a free $\mathbb{F}_2[U_{\boldsymbol{w}}]$-module generated by intersection points $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Define

$$CF^\infty(\mathcal{H}) := CF^-(\mathcal{H}) \otimes_{\mathbb{F}_2[U_{\boldsymbol{w}}]} \mathbb{F}_2[U_{\boldsymbol{w}}, U_{\boldsymbol{w}}^{-1}] \text{ and } CF^+(\mathcal{H}) := CF^\infty(\mathcal{H})/CF^-(\mathcal{H}).$$

To construct a differential on $CF^-(\mathcal{H})$, suppose $\mathcal{H}$ satisfies some extra admissibility conditions if $b_1(Y) > 0$ (*c.f.* [Zem19, Section 4.7]). Let $(J_s)_{s\in[0,1]}$ be an auxiliary path of almost complex structures on $\mathrm{Sym}^n\Sigma$ and let $\pi_2(\boldsymbol{x},\boldsymbol{y})$ be the set of homology classes of Whitney disks connecting intersection points $\boldsymbol{x}$ and $\boldsymbol{y}$ (*c.f.* [OS08a, Section 3.4]). For $\phi \in \pi_2(\boldsymbol{x},\boldsymbol{y})$, let $\mathcal{M}_{J_s}(\phi)$ be the moduli space of J_s-holomorphic maps $u : [0,1]\times\mathbb{R} \to \mathrm{Sym}^n\Sigma$ which represent ϕ. The moduli space $\mathcal{M}_{J_s}(\phi)$ has a natural action of $\mathbb{R}$, corresponding to reparametrization of the source. We write

$$\widehat{\mathcal{M}}_{J_s}(\phi) := \mathcal{M}_{J_s}(\psi)/\mathbb{R}.$$

For $\phi \in \pi_2(\boldsymbol{x},\boldsymbol{y})$, let $\mu(\phi)$ be the expected dimension of $\mathcal{M}_{J_s}(\phi)$ for generic J_s and let $n_{w_i}(\phi)$ be the algebraic intersection number of $\{w_i\}\times\mathrm{Sym}^{n-1}\Sigma$ and any representative of ϕ. Define

$$n_{\boldsymbol{w}}(\phi) := (n_{w_1}(\phi), \dots, n_{w_m}(\phi)).$$

For a generic path J_s, define the differential on $CF^-(\mathcal{H})$ by

$$\partial_{J_s}(\boldsymbol{x}) = \sum_{\boldsymbol{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\substack{\phi\in\pi_2(\boldsymbol{x},\boldsymbol{y})\\ \mu(\phi)=1}} \#\widehat{\mathcal{M}}_{J_s}(\phi) U_{\boldsymbol{w}}^{n_{\boldsymbol{w}}(\phi)} \cdot \boldsymbol{y},$$

extended linearly over $\mathbb{F}_2[U_{\boldsymbol{w}}]$. The differential ∂_{J_s} can be extended on $CF^\infty(\mathcal{H})$ and $CF^+(\mathcal{H})$ by tensoring with the identity map.

Lemma A.1.4 ([OS08a, Lemma 4.3]). *For a generic path J_s, the map ∂_{J_s} on $CF^\circ(\mathcal{H})$, where $\circ \in \{\infty,+,-\}$, satisfies*

$$\partial_{J_s} \circ \partial_{J_s} = 0.$$

For a disconnected multi-pointed 3-manifold $(Y,\boldsymbol{w}) = (Y_1,\boldsymbol{w_1}) \sqcup (Y_2,\boldsymbol{w_2})$, where Y_i is connected for $i = 1,2$, suppose $\mathcal{H}_i$ is an admissible multi-pointed Heegaard diagram of Y_i and suppose J_{s_i} are corresponding generic paths of almost complex structures. For $\circ \in \{\infty,+,-\}$, let the chain complex associated to $(Y, \boldsymbol{w})$ be

$$(CF^\circ(\mathcal{H}_1 \sqcup \mathcal{H}_2), \partial_{J_s}) := (CF^\circ(\mathcal{H}_1), \partial_{J_{s_1}}) \otimes_{\mathbb{F}_2} (CF^\circ(\mathcal{H}_2), \partial_{J_{s_2}}). \tag{A.1.1}$$

Remark A.1.5. In Zemke's original construction [Zem19, Section 4.3], one should choose colors for basepoints and graphs to achieve the functoriality of the TQFT. For basepoints with the same color, the corresponding U-variables should be the same. In above notations, we implicitly choose different colors for all basepoints so that the U-variable for each basepoint is different. This is to obtain the following relation on the homology level

$$H(CF^\circ(\mathcal{H}_1 \sqcup \mathcal{H}_2), \partial_{J_s}) = H(CF^\circ(\mathcal{H}_1), \partial_{J_{s_1}}) \otimes_{\mathbb{F}_2} H(CF^\circ(\mathcal{H}_1), \partial_{J_{s_2}}). \tag{A.1.2}$$

Note that in the construction of [HMZ18, Zem20], the colors of all basepoints are the same and all U-variables are identified as U, so (A.1.1) should be a tensor product over $\mathbb{F}_2[U]$ rather than $\mathbb{F}_2$ and (A.1.2) does not hold in general.

Remark A.1.6. Given a finite set of multi-pointed 3-manifolds and ribbon graph cobordisms, the chain complex $CF^-(\emptyset)$ is set to be $\mathbb{F}_2[U_{\boldsymbol{w}}]$, where $U_{\boldsymbol{w}}$ contains all U-variables associated to basepoints in the set. For any multi-pointed 3-manifold $(Y, \boldsymbol{w}')$ with $\boldsymbol{w}' \subset \boldsymbol{w}$ that is in the given set, the actual chain complex in the TQFT should be

$$CF^-(Y, \boldsymbol{w}') \otimes_{\mathbb{F}_2} \mathbb{F}_2[U_{\boldsymbol{w} \backslash \boldsymbol{w}'}].$$

In the statements of results in this paper, we always have $\boldsymbol{w}' = \boldsymbol{w}$ for any multi-pointed 3-manifold $(Y, \boldsymbol{w}')$. However, in the proof of those results (*e.g.* Lemma A.1.35 and Theorem A.1.30), we may have multi-pointed 3-manifold $(Y, \boldsymbol{w}')$ such that $\boldsymbol{w}' \neq \boldsymbol{w}$; see Remark A.1.36. Also, in the proof, the colors of basepoints may be different.

The chain homotopy type of $(CF^\circ(\mathcal{H}), \partial_{J_s})$ is independent of the choices of the admissible diagram $\mathcal{H}$ and the generic path J_s. Indeed, we have the following theorem about naturality.

Theorem A.1.7 ([Zem19, Proposition 4.6], see also [OS04d, JTZ21])**.** *Suppose that* $(Y, \boldsymbol{w})$ *is a multi-pointed 3-manifold. To each (admissible) pairs* $(\mathcal{H}, J)$ *and* $(\mathcal{H}', J')$*, there is a well-defined map*

$$\Psi_{(\mathcal{H},J)\to(\mathcal{H}',J')} : (CF^-(\mathcal{H}), \partial_J) \to CF^-(\mathcal{H}'), \partial_{J'}),$$

which is well-defined up to $\mathbb{F}_2[U_{\boldsymbol{w}}]$*-equivariant chain homotopy. Furthermore, the following holds.*

(1) If $(\mathcal{H},J)$*,* $(\mathcal{H}',J')$ *and* $(\mathcal{H}'',J'')$ *are three pairs, then there is a chain homotopy equivalence*

$$\Psi_{(\mathcal{H},J)\to(\mathcal{H}'',J'')} \simeq \Psi_{(\mathcal{H}',J')\to(\mathcal{H}'',J'')} \circ \Psi_{(\mathcal{H},J)\to(\mathcal{H}',J')}.$$

(2) $\Psi_{(\mathcal{H},J)\to(\mathcal{H},J)} \simeq \mathrm{id}_{(CF^-(\mathcal{H}),\partial_J)}$.

Moreover, similar results hold for CF^∞ *and* CF^+*.*

Convention. If it is not mentioned, chain homotopy means $\mathbb{F}_2[U_{\boldsymbol{w}}]$-equivariant chain homotopy.

Since all chain complexes discussed above can be decomposed into spin^c structures (*c.f.* [OS04d, Section 2.6]), we have the following definition.

Definition A.1.8. Suppose $(Y,\boldsymbol{w})$ is a multi-pointed 3-manifold and $\mathfrak{s} \in \mathrm{Spin}^c(Y)$. For $\circ \in \{\infty,+,-\}$, define $CF^\circ(Y,\boldsymbol{w},\mathfrak{s})$ to be the transitive system of chain complexes with canonical maps from Theorem A.1.7, with respect to $\mathfrak{s}$, and define $HF^\circ(Y,\boldsymbol{w},\mathfrak{s})$ to be the induced transitive system of homology groups.

For later use, we also define the completions of the chain complexes.

Definition A.1.9. Let $\mathbb{F}_2[[U_{\boldsymbol{w}}]]$ be the ring of formal power series of $U_{\boldsymbol{w}}$. For $\circ \in \{\infty,+,-\}$, define

$$\mathbf{CF}^\circ(Y,\boldsymbol{w},\mathfrak{s}) := CF^\circ(Y,\boldsymbol{w},\mathfrak{s}) \otimes_{\mathbb{F}_2[U_{\boldsymbol{w}}]} \mathbb{F}_2[[U_{\boldsymbol{w}}]].$$

Let $\mathbf{HF}^\circ(Y,\boldsymbol{w},\mathfrak{s})$ be the induced homology groups.

Convention. When omitting the module structure, we have $\mathbf{CF}^+(Y,\boldsymbol{w},\mathfrak{s}) = CF^+(Y,\boldsymbol{w},\mathfrak{s})$. Hence we do not distinguish them.

The advantage of the completions is that we have the following proposition.

Proposition A.1.10 ([MO17, Section 2], see also [OS04a, Lemma 2.3])**.** *If* $(Y,\boldsymbol{w})$ *is a multi-pointed 3-manifold and* $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ *on each component is nontorsion, then* $\mathbf{HF}^\infty(Y,\boldsymbol{w},\mathfrak{s}) = 0$.

Then the boundary map in the following long exact sequence induces a canonical isomorphism between $\mathbf{HF}^-(Y,\boldsymbol{w},\mathfrak{s})$ and $HF^+(Y,\boldsymbol{w},\mathfrak{s})$ for any nontorsion spin^c structure $\mathfrak{s}$.

Proposition A.1.11. *From the short exact sequence*

$$0 \to \mathbf{CF}^-(Y,\boldsymbol{w},\mathfrak{s}) \to \mathbf{CF}^\infty(Y,\boldsymbol{w},\mathfrak{s}) \to CF^+(Y,\boldsymbol{w},\mathfrak{s}) \to 0,$$

we have a long exact sequence

$$\cdots \to \mathbf{HF}^-(Y,\boldsymbol{w},\mathfrak{s}) \to \mathbf{HF}^\infty(Y,\boldsymbol{w},\mathfrak{s}) \to HF^+(Y,\boldsymbol{w},\mathfrak{s}) \to \cdots$$

We also have a long exact sequence for HF^-, HF^∞, and HF^+.

Definition A.1.12. Suppose $(Y,\boldsymbol{w})$ is a multi-pointed 3-manifold and $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ is a nontorsion spin^c structure. We write

$$HF(Y,\boldsymbol{w},\mathfrak{s}) = HF_{\mathrm{red}}(Y,\boldsymbol{w},\mathfrak{s}) := HF^+(Y,\boldsymbol{w},\mathfrak{s}) \cong \mathbf{HF}^-(Y,\boldsymbol{w},\mathfrak{s}).$$

A.1.2 Cobordism maps for restricted graph cobordisms

Theorem A.1.13 ([Zem19, Theorem A]). *Suppose $(W,\Gamma) : (Y_0,\boldsymbol{w}_0) \to (Y_1,\boldsymbol{w}_1)$ is a ribbon graph cobordism and $\mathfrak{s} \in \mathrm{Spin}^c(W)$. Then there are two chain maps*

$$F^A_{W,\Gamma,\mathfrak{s}}, F^B_{W,\Gamma,\mathfrak{s}} : CF^-(Y_0,\boldsymbol{w}_0,\mathfrak{s}|_{Y_0}) \to CF^-(Y_1,\boldsymbol{w}_1,\mathfrak{s}|_{Y_1}),$$

which are diffeomorphism invariants of (W,Γ), up to $\mathbb{F}_2[U_{\boldsymbol{w}}]$-equivariant chain homotopy.

Proposition A.1.14 ([Zem19, Theorem C]). *Suppose that (W,Γ) is a ribbon graph cobordism which decomposes as a composition $(W,\Gamma) = (W_2,\Gamma_2) \cup (W_1,\Gamma_1)$. If $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are spin^c structures on W_1 and W_2, respectively, then*

$$F^A_{W_2,\Gamma_2,\mathfrak{s}_2} \circ F^A_{W_1,\Gamma_1,\mathfrak{s}_1} \simeq \sum_{\substack{\mathfrak{s}\in\mathrm{Spin}^c(W)\\ \mathfrak{s}|_{W_2}=\mathfrak{s}_2\\ \mathfrak{s}|_{W_1}=\mathfrak{s}_1}} F^A_{W,\Gamma,\mathfrak{s}}.$$

A similar relation holds for $F^B_{W,\Gamma,\mathfrak{s}}$.

Since we will only consider restricted graph cobordisms, the map $F^A_{W,\Gamma,\mathfrak{s}}$ is chain homotopic to $F^B_{W,\Gamma,\mathfrak{s}}$. Hence we write $CF^-(W,\Gamma,\mathfrak{s})$ for the chain map and $HF^-(W,\Gamma,\mathfrak{s})$ for the induced map on the homology group. If Γ and $\mathfrak{s}$ are specified, we write $CF^-(W)$ and $HF^-(W)$ for simplicity, respectively. The chain maps on $CF^\infty, CF^+, \mathbf{CF}^-, \mathbf{CF}^\infty$ are obtained by tensoring with the identity maps, respectively. We use similar notations for these chain maps and the induced maps on homology groups. All maps are called **cobordism maps**.

For CF^-, the cobordism map is defined by the composition of the following maps.

- For 4-dimensional 1-, 2-, and 3-handle attachments away from the basepoints, we use the maps defined by Ozsváth and Szabó [OS06a].

- For 4-dimensional 0- and 4-handle attachments, or equivalently adding and removing a copy of S^3 with a single basepoint, respectively, we use the maps defined by the canonical isomorphism from the tensor product with $CF^-(S^3, w_0) \cong \mathbb{F}_2[U_0]$.
- For a ribbon graph cobordism $(Y \times [0,1], \Gamma)$, we project the graph into Y and use the **graph action map** defined in [Zem19, Section 7].

Remark A.1.15. For 4-dimensional 1-, 2-, and 3-handle attachments, Ozsváth and Szabó's original construction was for connected cobordisms between connected 3-manifolds. Zemke [Zem19, Section 8] extended the construction to cobordisms between possibly disconnected 3-manifolds. For 4-dimensional 0- and 4-handle attachments, the isomorphism is indeed

$$CF^-(Y \sqcup S^3, \boldsymbol{w} \cup \{w_0\}) \cong CF^-(Y, \boldsymbol{w}) \otimes_{\mathbb{F}_2} CF^-(S^3, w_0) \cong CF^-(Y, \boldsymbol{w}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[U_0].$$

The graph action map is obtained by the composition of maps associated to elementary graphs. The construction involves **free-stabilization maps** $S_w^{\pm}$ [Zem19, Section 6] and **relative homology maps** A_λ [Zem19, Section 5], where $S_w^{\pm}$ correspond to adding or removing a basepoint w and A_λ correspond to a path λ between two basepoints. When considering restricted graph cobordisms, we only need maps associated to 1-, 2-, 3-handle attachments.

Definition A.1.16. Suppose $\mathcal{H} = (\Sigma, \alpha, \beta, \boldsymbol{w})$ is a multi-pointed Heegaard diagram for a multi-pointed 3-manifold $(Y, \boldsymbol{w})$. Let $D \subset \Sigma \backslash (\alpha \cup \beta)$ be a small disk containing a new basepoint $w_0 \in \Sigma \backslash (\alpha \cup \beta)$. Let α_0 and β_0 be two simple closed curves on Σ bounding a disk containing w_0 and $|\alpha_0 \cap \beta_0| = 2$. Suppose θ^+ and θ^- are the higher and the lower graded intersection points, respectively. See Figure A.1. Consider the Heegaard diagram $\mathcal{H}' = (\Sigma, \alpha \cup \{\alpha_0\}, \beta \cup \{\beta_0\}, \boldsymbol{w} \cup \{w_0\})$, where α_0 and β_0 are in the region of a basepoint $z \in \boldsymbol{w}$.

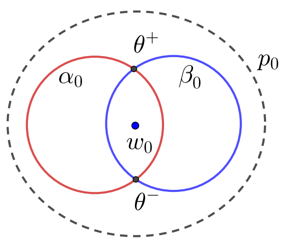

Figure A.1 Free-stabilization in a small disk D.

For appropriately chosen almost complex structures, define the **free-stabilization** maps $S_{w_0}^{\pm}$ by

$$S_{w_0}^{+}(\boldsymbol{x}) = \boldsymbol{x} \times \theta^+,$$

$$S^-_{w_0}(\boldsymbol{x}\times\theta^-)=\boldsymbol{x} \text{ and } S^-_{w_0}(\boldsymbol{x}\times\theta^+)=0.$$

Remark A.1.17. If we collapse ∂D to a point p_0, we obtain a doubly-pointed diagram on S^2 with two curves. Hence $\mathcal{H}'$ can be considered as the connected sum of $\mathcal{H}$ and $(S^2,\alpha_0,\beta_0,\{w_0,p_0\})$ at the basepoint z in $\mathcal{H}$ and the basepoint p_0 (*c.f.* [OS08a, Section 6.1]).

Proposition A.1.18 ([Zem19, Section 6 and Lemma 8.13])**.** *The maps $S^{\pm}_{w_0}$ in Definition A.1.16 determine well-defined chain maps on the level of transitive systems of chain complexes*

$$S^+_{w_0}: CF^-(Y,\boldsymbol{w})\to CF^-(Y,\boldsymbol{w}\cup\{w_0\}),$$

$$S^-_{w_0}: CF^-(Y,\boldsymbol{w}\cup\{w_0\})\to CF^-(Y,\boldsymbol{w}).$$

Moreover, they have the following properites.

(1) The maps $S^{\pm}_{w_0}$ commute with maps associated to 1-, 2-, and 3-handle attachments.

(2) For $\circ_1,\circ_2\in\{+,-\}$, we have $S^{\circ_1}_{w_1}S^{\circ_2}_{w_2}\simeq S^{\circ_2}_{w_2}S^{\circ_1}_{w_1}$.

Remark A.1.19. The free-stabilization maps can be regarded as ribbon graph cobordisms with $W=Y\times[0,1]$. The graphs are shown in Figure A.2. Alternatively, we can regard them as compositions of maps associated to handle attachments. The map $S^+_{w_2}$ is obtained by first attaching a 0-handle with an arc whose one endpoint is on the boundary, and the other is in the interior, and then attaching a 1-handle away from basepoints; see the left of Figure A.2. The map $S^-_{w_2}$ is obtained by first attaching a 3-handle and then a 4-handle with an arc similarly; see the right of Figure A.2.

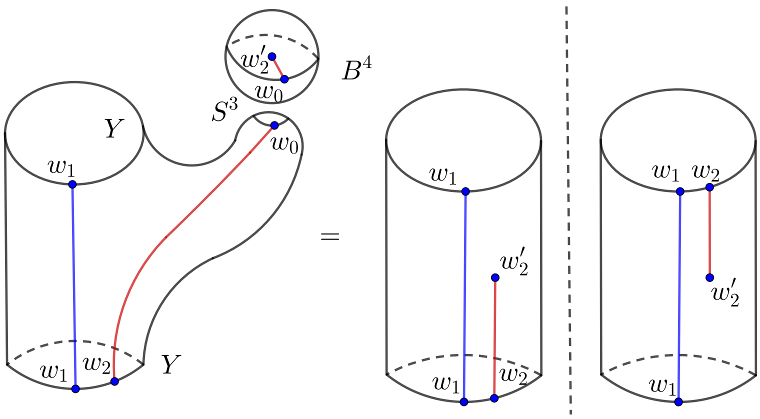

Figure A.2 Ribbon graph cobordisms related to free-stabilization maps.

Convention. All illustrations of cobordisms are from top to bottom.

We can calculate the effect of free-stabilization maps on the homology explicitly.

Proposition A.1.20 ([OS08a, Proposition 6.5])**.** *Consider the construction in Definition A.1.16. For suitable choices of almost complex structures, the chain complex $CF^-(\mathcal{H}')$ is identified with the mapping cone of the following map*

$$CF^-(\mathcal{H}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[U_0]\langle\theta^-\rangle \xrightarrow{U_0-U_1} CF^-(\mathcal{H}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[U_0]\langle\theta^+\rangle,$$

where U_1 corresponds to the basepoint in the original diagram $\mathcal{H}$ for the connected sum construction in Remark A.1.17.

Corollary A.1.21. *If $U_0 \neq U_1$ in Proposition A.1.20, i.e. the colors of corresponding basepoints are different (c.f. Remark A.1.5), then the map $S^+_{w_0}$ induces isomorphisms on HF° and $\mathbf{HF}^\circ$ for $\circ \in \{\infty, +, -\}$, and the map $S^-_{w_0}$ induces zero maps on all versions of Heegaard Floer homology.*

Proof. The arguments for $\circ \in \{\infty, -\}$ follows directly from Definition A.1.16, Proposition A.1.20, and definitions of Heegaard Floer homology groups. For $\circ = +$, note that the free-stabilization maps are compatible with the long exact sequence in Proposition A.1.11. Hence the behaviors of maps for $\circ \in \{\infty, -\}$ imply the behavior for $\circ = +$. □

The following proposition implies the choice of the basepoints is not important.

Proposition A.1.22 ([Zem19, Corollary 14.19 and Corollary F])**.** *Suppose $(Y, \boldsymbol{w})$ is a multi-pointed 3-manifold and $w_1 \in \boldsymbol{w}$. Then the $\pi_1(Y, w_1)$ action on $HF^-(Y, \boldsymbol{w})$ is always the identity map.*

Suppose $(Y_1, \boldsymbol{w}_1)$ and $(Y_2, \boldsymbol{w}_2)$ are two multi-pointed 3-manifolds with $|\boldsymbol{w}_1| = |\boldsymbol{w}_2|$. Suppose W is a cobordism from Y_1 to Y_2 such that the boundary of each component of W consists one component of $-Y_1$ and one component of Y_2. Suppose $\Gamma \subset W$ is a collection of paths connecting $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$. Then the cobordism map $HF^-(W, \Gamma)$ is independent of the choice of Γ. Moreover, if $W = Y \times I$, then $HF^-(W, \Gamma)$ is an isomorphism.

Similar results also hold for $HF^\infty, HF^+, \mathbf{HF}^-, \mathbf{HF}^\infty$.

From Corollary A.1.21 and Proposition A.1.22, we can define a transitive system of groups based on different choices of basepoints.

Definition A.1.23. Suppose Y is a closed, oriented 3-manifold and $\boldsymbol{w}_1, \boldsymbol{w}_2 \subset Y$ are two collections of basepoints in Y. Let $\boldsymbol{w}'_1 = \boldsymbol{w}_1 \backslash \boldsymbol{w}_2$ and $\boldsymbol{w}'_2 = \boldsymbol{w}_2 \backslash \boldsymbol{w}_1$. For $\circ \in \{\infty, +, -\}$, define

transition maps associated to $(\boldsymbol{w}_1, \boldsymbol{w}_2)$ as

$$\Psi^\circ_{\boldsymbol{w}_1\to\boldsymbol{w}_2} := \prod_{w\in\boldsymbol{w}'_1}(S^+_w)^{-1}\circ\prod_{w\in\boldsymbol{w}'_2}S^+_w \quad \text{on } HF^\circ \text{ and } \mathbf{HF}^\circ$$

where the products mean compositions. The order of maps is not important by the following lemma.

Lemma A.1.24. *Suppose Y is a closed, oriented 3-manifold and $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3 \subset Y$ are three collections of basepoints in Y. Suppose w is a basepoint in Y that is not in $\boldsymbol{w}_i$ for $i = 1, 2$. Then the following holds for transition maps.*

(1) $\Psi^\circ_{\boldsymbol{w}_i\to\boldsymbol{w}_j}$ is well-defined for $i, j \in \{1,2,3\}$, i.e., the composition is independent of the order of maps.

(2) $\Psi^\circ_{\boldsymbol{w}_i\to\boldsymbol{w}_j}$ is an isomorphism for $i, j \in \{1,2,3\}$.

(3) $\Psi^\circ_{\boldsymbol{w}_i\to\boldsymbol{w}_i} = \mathrm{id}$ for $i = 1,2,3$.

(4) $\Psi^\circ_{\boldsymbol{w}_2\to\boldsymbol{w}_3}\circ\Psi^\circ_{\boldsymbol{w}_1\to\boldsymbol{w}_2} = \Psi^\circ_{\boldsymbol{w}_1\to\boldsymbol{w}_3}$.

(5) $\Psi^\circ_{\boldsymbol{w}_1\cup\{w\}\to\boldsymbol{w}_2\cup\{w\}}\circ S^+_w = S^+_w\circ\Psi^\circ_{\boldsymbol{w}_1\to\boldsymbol{w}_2}$.

(6) $\Psi^\circ_{\boldsymbol{w}_1\to\boldsymbol{w}_2}\circ S^-_w = S^-_w\circ\Psi^\circ_{\boldsymbol{w}_1\cup\{w\}\to\boldsymbol{w}_2\cup\{w\}}$.

Proof. Terms (1), (4), (5) and (6) follow from term (2) of Proposition A.1.18. Note that maps in terms (5) are both isomorphisms and the maps in term (6) are both zero maps. Term (3) is trivial from the definition. Term (2) follows from Corollary A.1.21. □

Lemma A.1.25. *Suppose Y_1 and Y_2 are closed, oriented 3-manifolds and $\boldsymbol{w}_1, \boldsymbol{w}_2 \subset Y_1$, $\boldsymbol{w}_3, \boldsymbol{w}_4 \subset Y_2$ are collections of basepoints. Suppose W is a cobordism from Y_1 to Y_2 that is obtained from $Y_1 \times I$ by attaching 4-dimensional 1-, 2-, 3-handles away from all basepoints. Let $\Gamma_1 = \boldsymbol{w}_1 \times I$ be the induced graph in W and suppose $\boldsymbol{w}_3$ is the image of $\boldsymbol{w}_1 \times \{1\}$. The cobordism W can also be obtained from $-Y_2 \times I$ by attaching handles away from basepoints and let $\Gamma_2 = \boldsymbol{w}_4 \times I$. Suppose the image of $\boldsymbol{w}_4$ is $\boldsymbol{w}_2$. Then we have a commutative diagram*

$$\begin{array}{ccc} HF^-(Y_1,\boldsymbol{w}_1) & \xrightarrow{HF^-(W,\Gamma_1)} & HF^-(Y_2,\boldsymbol{w}_3) \\ \Big\downarrow{\scriptstyle \Psi^-_{\boldsymbol{w}_1\to\boldsymbol{w}_2}} & & \Big\downarrow{\scriptstyle \Psi^-_{\boldsymbol{w}_3\to\boldsymbol{w}_4}} \\ HF^-(Y_1,\boldsymbol{w}_2) & \xrightarrow{HF^-(W,\Gamma_2)} & HF^-(Y_2,\boldsymbol{w}_4) \end{array}$$

Similar commutative diagrams hold for $\mathbf{HF}^-$ and HF^+.

Proof. This follows from term (1) of Proposition A.1.18. □

Theorem A.1.26. *Suppose Y is a closed, oriented 3-manifold. Then groups $HF^-(Y,\boldsymbol{w})$ for all $\boldsymbol{w}\subset Y$ and transition maps $\Psi^-_{\boldsymbol{w}_1\to\boldsymbol{w}_2}$ for all $\boldsymbol{w}_1,\boldsymbol{w}_2\subset Y$ form a transitive system, which is denoted by $HF^-(Y)$. Moreover, suppose (W,Γ) is a restricted graph cobordism from $(Y_1,\boldsymbol{w}_1)$ to $(Y_2,\boldsymbol{w}_2)$. Then $HF^-(W,\Gamma)$ induces a well-defined map from $HF^-(Y_1)$ to $HF^-(Y_2)$, which is independent of the choice of the restricted graph Γ and denoted by $HF^-(W)$.*

Similar arguments hold for infinity and plus versions of Heegaard Floer homology groups.

Proof. The well-definedness of $HF^-(Y)$ and $HF^-(W,\Gamma)$ follows from Lemma A.1.24 and Lemma A.1.25. Note that the restricted graph cobordism is a composition of maps associated to 1-, 2-, 3-handle attachments. Then the independence of Γ follows from the functoriality of the map associated to a ribbon graph cobordism. The proofs for infinity and plus versions of Heegaard Floer homology groups are similar. □

Remark A.1.27. Groups and maps in Theorem A.1.26 also split into spin^c structures. Suppose $\mathfrak{s}\in \mathrm{Spin}^c(W)$ is a nontorsion spin^c structure which restricts to nontorsion spin^c structure $\mathfrak{s}_i$ on Y_i for $i=1,2$. Then $\mathbf{HF}^-(Y_i,\mathfrak{s}_i)$ and $HF^+(Y_i,\mathfrak{s}_i)$ are canonically identified by the boundary map in Proposition A.1.11. Moreover, the maps $\mathbf{HF}^-(W,\mathfrak{s})$ and $HF^+(W,\mathfrak{s})$ are the same under this identification. We write the map as $HF(W,\mathfrak{s})$.

A.1.3 Floer's excision theorem

Note that the proofs of Theorem 2.3.16 and Theorem 2.3.20 (*c.f.* [BS15, Li19]) both involve Floer's excision theorem in an essential way. In this subsection, we follow Kronheimer-Mrowka's idea in [KM10b, Section 3] to prove an excision theorem for Heegaard Floer theory. Though for Heegaard Floer theory, we need to modify the proof to fit the settings of multi-basepoints 3-manifolds and ribbon graph cobordisms.

Let Y be a closed, oriented 3-manifold, of either one or two components. In the latter case, let Y_1 and Y_2 be two components of Y. Let Σ_1 and Σ_2 be two closed, connected, oriented surfaces in Y with $g(\Sigma_1)=g(\Sigma_2)$. If Y has two components, suppose Σ_i is a non-separating surface in Y_i for $i=1,2$. If Y is connected, suppose Σ_1 and Σ_2 represent independent homology classes. In either case, let $F=\Sigma_1\cup\Sigma_2$. Let h be an orientation-preserving diffeomorphism from Σ_1 to Σ_2.

We construct a new manifold $\widetilde{Y}$ as follows. Let Y' be obtained from Y by cutting along Σ. Then

$$\partial Y'=\Sigma_1\cup(-\Sigma_1)\cup\Sigma_2\cup(-\Sigma_2).$$

If Y has two components, then we have $Y' = Y_1' \cup Y_2'$, where Y_i' is obtained from Y_i by cutting along Σ_i for $i = 1,2$. Let $\widetilde{Y}$ be obtained from Y' by gluing the boundary component Σ_1 to the boundary component $-\Sigma_2$ and gluing Σ_2 to $-\Sigma_1$, using the diffeomorphism of h in both cases; see Figure A.3 for the case that Y has two components.

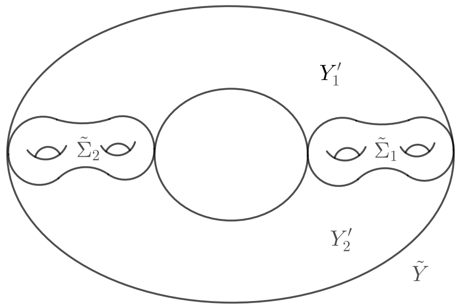

Figure A.3 Construction of $\widetilde{Y}$.

In either case, $\widetilde{Y}$ is connected. Let $\widetilde{\Sigma}_i$ be the image of Σ_i in $\widetilde{Y}$ for $i = 1,2$ and let $\widetilde{F} = \widetilde{\Sigma}_1 \cup \widetilde{\Sigma}_2$.

Definition A.1.28. Suppose Y is a closed, oriented 3-manifold and $F \subset Y$ is a closed, oriented surface. Let F_i for $i = 1,\dots,m$ be the components of F. Suppose further that $g(F_i) \geq 2$ and any component of Y contains at least one component of F. Let $\mathrm{Spin}^c(Y|F)$ denote the set of spin^c structures $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ satisfying

$$\langle c_1(\mathfrak{s}), F_i\rangle = 2g(F_i) - 2 \text{ for any } F_i. \tag{A.1.3}$$

Define

$$HF(Y|F) := \bigoplus_{\mathfrak{s}\in \mathrm{Spin}^c(Y|F)} HF(Y,\mathfrak{s}). \tag{A.1.4}$$

Suppose (W,Γ) is a restricted graph cobordism and $G \subset W$ is a closed, oriented surface. Let G_i for $i = 1,\dots,n$ be components of G. Suppose further that $g(G_i) \geq 2$ and any component of W contains at least one component of G. Let $\mathrm{Spin}^c(W|G)$ denote the set of spin^c structures $\mathfrak{s} \in \mathrm{Spin}^c(W)$ satisfying similar conditions in (A.1.3) by replacing F_i by G_i. Define

$$HF^-(W,\Gamma|G) := \sum_{\mathfrak{s}\in \mathrm{Spin}^c(W|G)} HF^-(W,\Gamma,\mathfrak{s}).$$

Let $HF^+(W,\Gamma|G)$, $\mathbf{HF}^-(W,\Gamma|G)$ and $HF(W,\Gamma|G)$ be defined similarly. We also denote the corresponding map on the chain level by replacing HF by CF.

Remark A.1.29. All spin^c structures in $\mathrm{Spin}^c(Y|F)$ are nontorsion, so $HF(Y,\mathfrak{s})$ is well-defined.

The following is the main theorem of this subsection.

Theorem A.1.30 (Floer's excision theorem)**.** *Consider Y and $\widetilde{Y}$ constructed as above. If $g(\Sigma_1) = g(\Sigma_2) \geq 2$, then there is an isomorphism*

$$HF(Y|F) \cong HF(\widetilde{Y}|\widetilde{F}).$$

Moreover, this isomorphism and its inverse are induced by restricted graph cobordisms.

Before proving the main theorem, we introduce some lemmas analogous to results in monopole theory (*c.f.* [KM10b, Lemma 2.2, Proposition 2.5 and Lemma 4.7])

Lemma A.1.31 ([Lek13, Theorem 16 and Corollary 17], see also [OS04a, Theorem 5.2])**.** *Let $Y \to S^1$ be a fibred 3-manifold whose fibre F is a closed, connected, oriented surface with $g = g(F) \geq 2$. Then $\mathbf{CF}^-(Y|F)$ is chain homotopic to the chain complex*

$$0 \to \mathbb{F}_2[[U_0]]\langle x\rangle \xrightarrow{U_0} \mathbb{F}_2[[U_0]]\langle y\rangle \to 0. \tag{A.1.5}$$

Moreover, there is a unique $\mathfrak{s}_0 \in \mathrm{Spin}^c(Y|R)$ so that $HF(Y, \mathfrak{s}_0) \neq 0$ and we have

$$HF(Y|F) = HF(Y, \mathfrak{s}_0) \cong \mathbb{F}_2.$$

Remark A.1.32. Indeed, for Y in Lemma A.1.31, we can construct a *weakly* admissible Heegaard diagram $\mathcal{H}$ for the singly-pointed 3-manifold (Y, w) so that $\mathbf{CF}^-(\mathcal{H}, \mathfrak{s}_0)$ is generated by $8g$ generators $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{4g}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{4g}$ and

$$\partial \boldsymbol{x}_1 = U_0 \boldsymbol{y}_1, \partial \boldsymbol{x}_j = \boldsymbol{y}_j, \text{ and } \partial \boldsymbol{y}_k = 0 \text{ for } j > 1, k \geq 1.$$

The reason to use $\mathbf{CF}^-$ rather than CF^- is because the computation of CF^- is based on *strongly* admissible Heegaard diagram.

Lemma A.1.33. *Suppose $Y = \Sigma \times S^1$ such that $\Sigma = \Sigma \times \{1\} \subset Y$ is a closed, connected, oriented surface with $g(\Sigma) \geq 2$. Suppose $w_0 \in S^3$ and $w \in Y$ are basepoints. Let W be obtained from $\Sigma \times D^2$ by removing a 4-ball, considered as a cobordism from S^3 to Y. Let $\Gamma \subset W$ be any path connecting w_0 to w. Then the map*

$$\mathbf{HF}^-(W, \Gamma|\Sigma) : \mathbb{F}_2[[U_0]] \cong \mathbf{HF}^-(S^3, w_0) \to HF(Y, w|\Sigma) \cong \mathbb{F}_2 \tag{A.1.6}$$

is nonzero.

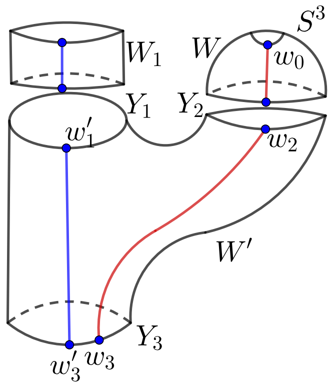

Figure A.4 Nontrivial cobordism map from composition.

Proof. Suppose P is 2-dimensional pair of pants as shown in Figure A.4. Consider $W' = \Sigma \times P$ as a cobordism from $Y_1 \sqcup Y_2$ to Y_3, where $Y_i \cong Y$ for $i = 1,2,3$. Suppose w' is another basepoint in Y. Let w_i and w_i' be the images of w and w' in Y_i for $i = 1,2,3$. Let $\Gamma' \subset W'$ be a collection of two paths γ_1 and γ_2, where γ_1 connects w_1' to w_3' and γ_2 connects w_2 to w_3.

Let $(W_1, \Gamma_1) = (Y_1 \times I, w' \times I)$ be the product cobordism. Suppose $\Sigma_i \subset Y_i$ is the image of $\Sigma \subset Y$ for $i = 1,2,3$. Consider the composition of the cobordism maps

$$\mathbf{HF}^-(W', \Gamma' | \Sigma_1 \cup \Sigma_2 \cup \Sigma_3) \circ \mathbf{HF}^-(W_1 \sqcup W, \Gamma_1 \sqcup \Gamma | \Sigma_1 \cup \Sigma_2) : \\ HF(Y_1, w_1' | \Sigma_1) \otimes_{\mathbb{F}_2} \mathbf{HF}^-(S^3, w_0) \to HF(Y_3, \{w_3, w_3'\} | \Sigma_3).$$

After filling the S^3 component by a 4-ball, or equivalently composing it with the map associated to a 0-handle attachment, we obtain the free-stabilization map S_w^+ (*c.f.* Remark A.1.19). By Corollary A.1.21, the resulting map is an isomorphism

$$HF(Y_1, w_1' | \Sigma_1) \cong HF(Y_3, \{w_3, w_3'\} | \Sigma_3).$$

Since

$$\mathbf{HF}^-(W_1 \sqcup W, \Gamma_1 \sqcup \Gamma | \Sigma_1 \cup \Sigma_2) = \mathbf{HF}^-(W_1, \Gamma_1 | \Sigma_1) \otimes_{\mathbb{F}_2} \mathbf{HF}^-(W, \Gamma | \Sigma_2),$$

and $\mathbf{HF}^-(W_1 | \Sigma_1)$ is the identity map, we know $\mathbf{HF}^-(W | \Sigma_2)$ is nonzero.

□

Corollary A.1.34. *On the chain level of (A.1.6), the cobordism map* $\mathbf{CF}^-(W, \Gamma | \Sigma)$ *sends the generator of* $\mathbf{CF}^-(S^3, w_0) \cong \mathbb{F}_2[[U_0]]$ *to the generator of second copy of* $\mathbb{F}_2[[U_0]]$ *in (A.1.5).*

Proof. The map in the statement is the only $\mathbb{F}_2[U_0]$-equivariant chain map that induces a nonzero map on the homology. □

The proof of the following lemma is due to Ian Zemke.

Lemma A.1.35. *Let $Y = \Sigma \times S^1$ and let $W_1 \cong Y \times I$ be a cobordism from $\emptyset$ to $Y \sqcup (-Y)$. Let $w_1 \in Y, w_2 \in (-Y), w_1', w_2' \in W_1$ and let $\Gamma_1 \subset W_1$ consist of two paths whose enpoints are w_i and w_i' for $i = 1, 2$, as shown in the left subfigure of Figure A.5. Let $W_2 \cong \Sigma \times D^2 \sqcup (-\Sigma \times D^2)$ be another cobordism from $\emptyset$ to $Y \sqcup (-Y)$ and let $\Gamma_2 \subset W_2$ be obtained from two copies of the cobordism in Lemma A.1.33 associated to Σ and $-\Sigma$ by filling the S^3 components by 4-balls (c.f. Remark A.1.19), as shown in the right subfigure of Figure A.5. Then we have*

$$\mathbf{CF}^-(W_1, \Gamma_1 | \Sigma \sqcup (-\Sigma)) \simeq \mathbf{CF}^-(W_2, \Gamma_2 | \Sigma \sqcup (-\Sigma)) : \mathbf{CF}^-(\emptyset) \to \mathbf{CF}^-(Y \sqcup (-Y), \{w_1, w_2\} | \Sigma \sqcup (-\Sigma)). \tag{A.1.7}$$

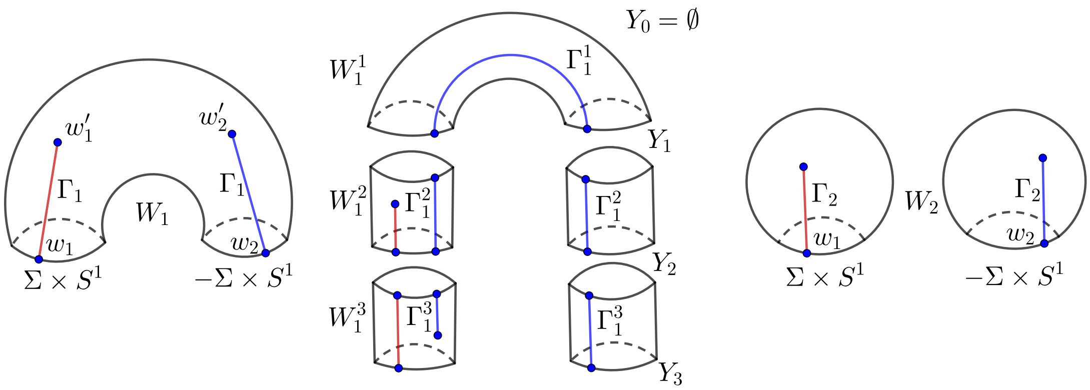

Figure A.5 Ribbon graph cobordisms (W_1, Γ_1) and (W_2, Γ_2).

Proof. Set $\mathcal{R} = \mathbb{F}_2[[U_1, U_2]]$. By Remark A.1.5, we implicitly choose w_1 and w_2 to have different colors and then

$$\mathbf{CF}^-(Y \sqcup (-Y), \{w_1, w_2\} | \Sigma \sqcup (-\Sigma)) := \mathbf{CF}^-(Y|\Sigma) \otimes_{\mathbb{F}_2} \mathbf{CF}^-(-Y| - \Sigma).$$

By Remark A.1.6, we have $\mathbf{CF}^-(\emptyset) = \mathcal{R}$. By TQFT property in [Zem19], we have a canonical chain isomorphism

$$\mathbf{CF}^-(-Y, w_2| - \Sigma) \cong \mathbf{CF}^-(Y, w_2|\Sigma)^\vee := \mathrm{Hom}_{\mathcal{R}}(\mathbf{CF}^-(Y, w_2|\Sigma), \mathcal{R}).$$

Then by Lemma A.1.31, we have

$$\mathbf{CF}^-(Y\sqcup(-Y),\{w_1,w_2\}|\Sigma\sqcup(-\Sigma))\simeq \begin{array}{ccc} \mathcal{R}\langle x\otimes y^\vee\rangle & \xrightarrow{U_1} & \mathcal{R}\langle x\otimes x^\vee\rangle \\ \downarrow{\scriptstyle U_2} & & \downarrow{\scriptstyle U_2} \\ \mathcal{R}\langle y\otimes y^\vee\rangle & \xrightarrow{U_1} & \mathcal{R}\langle y\otimes x^\vee\rangle \end{array} \tag{A.1.8}$$

where $x^\vee$ and $y^\vee$ are duals of x and y, respectively. By Corollary A.1.34, we know $\mathbf{CF}^-(W_2,\Gamma_2|\Sigma\sqcup(-\Sigma))$ sends the generator of $\mathbf{CF}^-(\emptyset)$ to $y\otimes x^\vee$ in (A.1.8).

By Proposition A.1.14, we compute $\mathbf{CF}^-(W_1,\Gamma_1|\Sigma\sqcup(-\Sigma))$ by decomposing (W_1,Γ_1) into three parts $(W_1^i,\Gamma_1^i):(Y_{i-1},\boldsymbol{w}_{i-1})\to(Y_i,\boldsymbol{w}_i)$ for $i=1,2,3$ as shown in the middle subfigure of Figure A.5. Note that $(Y_0,\boldsymbol{w}_0)=\emptyset$. Let F be the images of $\Sigma\sqcup(-\Sigma)$.

First, we compute $\mathbf{CF}^-(W_1^1,\Gamma_1^1|F)$. Since the two basepoints in $\boldsymbol{w}_1$ have the same color (also the same as w_2), we have

$$\mathbf{CF}^-(Y_1,\boldsymbol{w}_1|F)\simeq \begin{array}{ccc} \mathcal{R}\langle x\otimes y^\vee\rangle & \xrightarrow{U_2} & \mathcal{R}\langle x\otimes x^\vee\rangle \\ \downarrow{\scriptstyle U_2} & & \downarrow{\scriptstyle U_2} \\ \mathcal{R}\langle y\otimes y^\vee\rangle & \xrightarrow{U_2} & \mathcal{R}\langle y\otimes x^\vee\rangle \end{array} \tag{A.1.9}$$

From Zemke's calculation [Zem20, Theorem 1.7], the cobordism map $\mathbf{CF}^-(W_1^1,\Gamma_1^1|F)$ is the canonical cotrace map, *i.e.*, it sends the generator of $\mathbf{CF}^-(\emptyset)=\mathcal{R}$ to $x\otimes x^\vee+y\otimes y^\vee$. Note that the original calculation is for CF^- but it is easy to extend the result to $\mathbf{CF}^-$.

Remark A.1.36. Though we only have one color in $\boldsymbol{w}_1$, we use $\mathcal{R}$ rather than $\mathbb{F}_2[[U_2]]$ in (A.1.9) to achieve the functoriality (*c.f.* Remark A.1.6). Thus, when applying Proposition A.1.20 in the following computation, we do not need to add another U-variable.

Second, we compute $\mathbf{CF}^-(W_1^2,\Gamma_1^2|F)$. Note that the left component of (W_1^2,Γ_1^2) corresponds to the free-stabilization map $S^+_{w_1}$ and the right component is just the identity map. By Proposition A.1.20, the chain complex $\mathbf{CF}^-(Y_2,\boldsymbol{w}_2|F)$ is chain homotopic to the mapping cone of

$$\left(\begin{array}{ccc} \mathcal{R}\langle x\otimes y^\vee\otimes\theta^-\rangle & \xrightarrow{U_2} & \mathcal{R}\langle x\otimes x^\vee\otimes\theta^-\rangle \\ \downarrow{\scriptstyle U_2} & & \downarrow{\scriptstyle U_2} \\ \mathcal{R}\langle y\otimes y^\vee\otimes\theta^-\rangle & \xrightarrow{U_2} & \mathcal{R}\langle y\otimes x^\vee\otimes\theta^-\rangle \end{array}\right) \xrightarrow{U_1-U_2} \left(\begin{array}{ccc} \mathcal{R}\langle x\otimes y^\vee\otimes\theta^+\rangle & \xrightarrow{U_2} & \mathcal{R}\langle x\otimes x^\vee\otimes\theta^+\rangle \\ \downarrow{\scriptstyle U_2} & & \downarrow{\scriptstyle U_2} \\ \mathcal{R}\langle y\otimes y^\vee\otimes\theta^+\rangle & \xrightarrow{U_2} & \mathcal{R}\langle y\otimes x^\vee\otimes\theta^+\rangle \end{array}\right) \tag{A.1.10}$$

where $u\otimes v\otimes\theta^\pm$ for $u\in\{x,x^\vee\},y\in\{y,y^\vee\}$ represents $(u\times\theta^\pm)\otimes v$. Then $\mathbf{CF}^-(W_1^2,\Gamma_1^2|F)$ sends any generator $u\otimes v$ to $u\otimes v\otimes\theta^+$ in (A.1.10).

Third, we compute $\mathbf{CF}^-(W_1^3,\Gamma_1^3|F)$. Note that the left component of (W_1^3,Γ_1^3) corresponds to the free-stabilization map $S_{w_2}^-$ and the right component is just the identity map. Also by Proposition A.1.20, the chain complex $\mathbf{CF}^-(Y_2,\boldsymbol{w}_2|F)$ is chain homotopic to the mapping cone of

$$\left(\begin{array}{ccc}\mathcal{R}\langle x\otimes y^\vee\otimes\theta^-\rangle & \xrightarrow{U_1} & \mathcal{R}\langle x\otimes x^\vee\otimes\theta^-\rangle\\ \downarrow{\scriptstyle U_2} & & \downarrow{\scriptstyle U_2}\\ \mathcal{R}\langle y\otimes y^\vee\otimes\theta^-\rangle & \xrightarrow{U_1} & \mathcal{R}\langle y\otimes x^\vee\otimes\theta^-\rangle\end{array}\right)\xrightarrow{U_2-U_1}\left(\begin{array}{ccc}\mathcal{R}\langle x\otimes y^\vee\otimes\theta^+\rangle & \xrightarrow{U_1} & \mathcal{R}\langle x\otimes x^\vee\otimes\theta^+\rangle\\ \downarrow{\scriptstyle U_2} & & \downarrow{\scriptstyle U_2}\\ \mathcal{R}\langle y\otimes y^\vee\otimes\theta^+\rangle & \xrightarrow{U_1} & \mathcal{R}\langle y\otimes x^\vee\otimes\theta^+\rangle\end{array}\right) \tag{A.1.11}$$

Then $\mathbf{CF}^-(W_1^3,\Gamma_1^3|F)$ sends $u\otimes v\otimes\theta^-$ to $u\otimes v$ in (A.1.8) and sends $u\otimes v\otimes\theta^+$ to 0 for $u\in\{x,x^\vee\},y\in\{y,y^\vee\}$.

To compute the composition, we need to find the explicit chain homotopy between the above two mapping cones (A.1.10) and (A.1.11), which is calculated by Zemke [Zem19, Theorem 14.1]. Since we only care about the image of $\mathbf{CF}^-(\emptyset)$, we only need to calculate the image of $*$ map in [Zem19, (14.3)] (from the target in (A.1.10) to the source in (A.1.11))

$$(\Psi_{\alpha\to\alpha'}^{\beta'})_{U_w}^{U_z\to U_{w'}}\circ\left(\sum_{i,j\geq 0}U_w^iU_{w'}^j(\partial_{i+j+1})_{U_w,U_{w'}}\right)\circ(\Psi_\alpha^{\beta\to\beta'})_{U_{w'}}^{U_z\to U_w} \tag{A.1.12}$$

for the element

$$x\otimes x^\vee\otimes\theta^+ + y\otimes y^\vee\otimes\theta^+ \tag{A.1.13}$$

in (A.1.10). In (A.1.12), we have $z\in Y_1$ for the connected sum construction in Remark A.1.17, $w=w_2,w'=w_1,U_w=U_2,U_{w'}=U_1$ and α',β' being small isotopies of α,β, respectively. The differential ∂_k comes from

$$\partial=\sum_{k\in\mathbb{N}}U_z^k\partial_k, \tag{A.1.14}$$

where ∂ is the differential in

$$\mathbf{CF}^-(Y_1,\{z,w_2\}|\Sigma\sqcup(-\Sigma))\simeq\begin{array}{ccc}\mathcal{R}\langle x\otimes y^\vee\rangle & \xrightarrow{U_z} & \mathcal{R}\langle x\otimes x^\vee\rangle\\ \downarrow{\scriptstyle U_2} & & \downarrow{\scriptstyle U_2}\\ \mathcal{R}\langle y\otimes y^\vee\rangle & \xrightarrow{U_z} & \mathcal{R}\langle y\otimes x^\vee\rangle\end{array} \tag{A.1.15}$$

For a map f, the notation $(f)^{U_z\to U_w}$ means we replace U_z by U_w in the image of f and the notation $(f)_{U_w}$ means tensoring f with the identity map in $\mathbb{F}_2[U_w]$.

Since the element (A.1.13) has no U-power, the transition maps $(\Psi_{\alpha\to\alpha'}^{\beta'})_{U_w}^{U_z\to U_{w'}}$ and $(\Psi_\alpha^{\beta\to\beta'})_{U_{w'}}^{U_z\to U_w}$ can be regarded as identity maps. By (A.1.14) and (A.1.15), we know $\partial_k=0$

for $k \geq 1$ and ∂_1 sends $x \otimes x^\vee$ to 0 and sends $y \otimes y^\vee$ to $y \otimes x^\vee$. Hence the $*$ map (A.1.12) sends the element (A.1.13) to $y \otimes x^\vee \otimes \theta^-$ in (A.1.11).

Thus, by composing three cobordism maps and up to chain homotopy, we show that $\mathbf{CF}^-(W_1, \Gamma_1|\Sigma \sqcup (-\Sigma))$ also sends the generator of $\mathbf{CF}^-(\emptyset) = \mathcal{R}$ to $y \otimes x^\vee$ in (A.1.8). □

Now we start to prove the main theorem of this subsection. The basic idea is from Kronheimer-Mrowka [KM10b, Section 3.2], which originally came from Floer's work [Flo90], where he dealt with the excision theorem in instanton theory for the genus one case.

Proof of Theorem A.1.30. **Step 1**. We construct a cobordism W from $\widetilde{Y}$ to Y and a cobordism $\bar{W}$ from Y to $\widetilde{Y}$.

Recall that Y' is obtained from Y by cutting along Σ_1 and Σ_2 and we have

$$\partial Y' = \Sigma_1 \cup (-\Sigma_1) \cup \Sigma_2 \cup (-\Sigma_2).$$

Suppose P_1 is a saddle surface, which can be regarded as a submanifold of a pair of pants with one boundary component on the top and two boundary components at the bottom; see the left subfigure of Figure A.6. Suppose

$$\partial P_1 = \lambda_1 \cup \lambda_2 \cup \mu_1 \cup \mu_2 \cup \eta_{1,1} \cup \eta_{1,2} \cup \eta_{2,1} \cup \eta_{2,2},$$

where λ_1 and λ_2 are two arcs in the top boundary component of the pair of pants, μ_1 and μ_2 are two arcs in the bottom boundary components of the pair of pants, and $\eta_{i,j}$ is the arc connecting λ_i and μ_j for $i, j \in \{1,2\}$.

Suppose $\Sigma \cong \Sigma_1 \cong \Sigma_2$. Note that we have fixed a diffeomorphism h from Σ_1 to Σ_2. Suppose h' is an orientation-preserving diffeomorphism from Σ to Σ_1. Let W be the union

$$P_1 \times \Sigma \cup Y' \times I,$$

where $\eta_{1,1} \times \Sigma$ is glued to $\Sigma_1 \times I$, $\eta_{2,1} \times \Sigma$ is glued to $-\Sigma_1 \times I$, $\eta_{2,2} \times \Sigma$ is glued to $\Sigma_2 \times I$, and $\eta_{1,2} \times \Sigma$ is glued to $-\Sigma_2 \times I$, using h' and $h \circ h'$, respectively. Figure A.6 illustrates the case that Y' has two components Y_1' and Y_2'. By the construction of $\widetilde{Y}$, the resulting manifold W is a cobordism from $\widetilde{Y}$ to Y.

The cobordism $\bar{W}$ is constructed similarly. Let P_2 be another saddle surface and let $\bar{W}$ be obtained by gluing $P_2 \times \Sigma$ and $Y' \times I$ as shown in the right subfigure of Figure A.6.

Step 2. For some restricted graph Γ_A and some surface G_A in $W_A = \bar{W} \cup_{\widetilde{Y}} W$, we show the cobordism map

$$HF(W_A, \Gamma_A|G_A) := HF^+(W_A, \Gamma_A|G_A) = \mathbf{HF}^-(W_A, \Gamma_A|G_A)$$

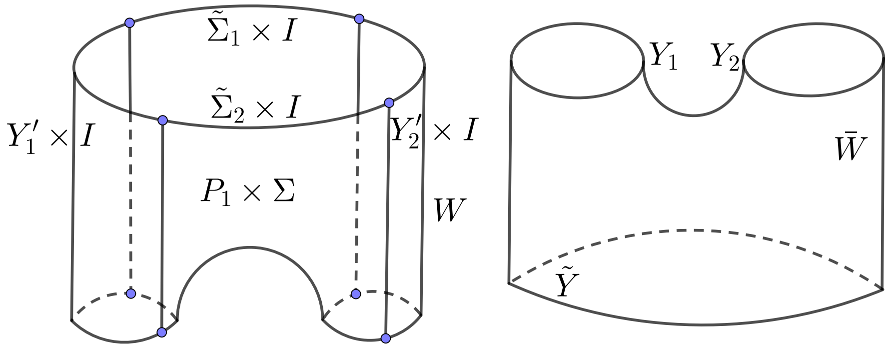

Figure A.6 Cobordisms W and $\bar{W}$.

induces the identity map on

$$HF(Y|F) := HF^+(Y|F) \cong \mathbf{HF}^-(Y|F).$$

We prove this for the case that Y has two components Y_1 and Y_2. The proof for the case that Y is connected is similar. For $i = 1,2$, let $w_i \in Y_i$ be basepoints and let $\Gamma_A \subset W_A$ consist of paths connecting basepoints w_i in different ends of W_A; see the left subfigure of Figure A.7. Suppose W'_A is diffeomorphic to W_A but drawn in a different position and suppose $\Gamma'_A \subset W'_A$ is obtained from Γ_A by adding an arc to each path and choosing any ordering for the vertex with valence 3; see the middle subfigure of Figure A.7. By [Zem19, Section 11.2], the ribbon graph cobordisms (W_A, Γ_A) and (W'_A, Γ'_A) induce the same cobordism map. Suppose $Y_A \cong \Sigma \times S^1 \subset W_A$ is the manifold in the neck of W'_A. We know a neighborhood $N(Y_A)$ is diffeomorphic to $Y_0 \times I$. Let G_A consist of the images of Σ in ∂W_A and $\partial N(Y_0)$.

By Proposition A.1.14, we can decompose (W'_A, Γ'_A) into two parts as shown in the left subfigure of Figure A.8 and compute $HF(W_A, \Gamma_A|G_A)$ by composition of two cobordism maps. The first part has three components corresponding to $Y_1 \times I, N(Y_A)$, and $Y_2 \times I$, respectively. By Lemma A.1.35, we can replace the component corresponding to $N(Y_A)$ by two components corresponding to $\Sigma \times D^2 \sqcup (-\Sigma \times D^2)$ in the right subfigure of Figure A.5. Then we know the cobordism map $HF(W_A, \Gamma'_A|G_A)$ is the same as $HF(W''_A, \Gamma''|G_A)$, where (W''_A, Γ'') is the ribbon graph cobordism in the right subfigure of Figure A.8. By [Zem19, Section 11.2], we can remove the arcs of Γ'' in the interior of the cobordism W''_A. Then we know $HF(W''_A, \Gamma''_A|G_A)$ is the identity map because

$$(W''_A, \Gamma''_A) \cong ((Y_1 \sqcup Y_2) \times I, (w_1 \sqcup w_2) \times I).$$

Thus, the cobordism map $HF(W_A, \Gamma|G_A)$ is the identity map.

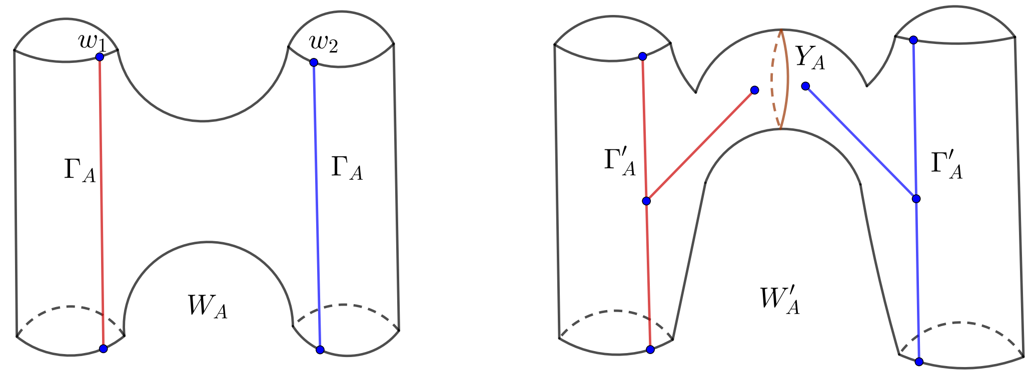

Figure A.7 Ribbon graph cobordisms (W_A, Γ_A) and (W'_A, Γ'_A).

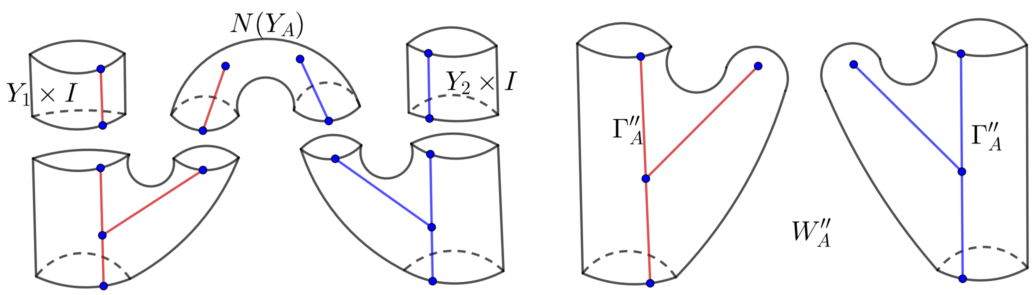

Figure A.8 Ribbon graph cobordisms (W'_A, Γ'_A) and (W''_A, Γ''_A).

Step 3. For some restricted graph Γ_B and some surface G_B in $W_B = W \cup_Y \bar{W}$, we show the cobordism map

$$HF(W_B, \Gamma_B | G_B) := HF^+(W_B, \Gamma_B | G_B) = \mathbf{HF}^-(W_B, \Gamma_B | G_B)$$

induces the identity map on

$$HF(\widetilde{Y}|\widetilde{F}) := HF^+(\widetilde{Y}|\widetilde{F}) \cong \mathbf{HF}^-(\widetilde{Y}|\widetilde{F}).$$

We prove this for the case that Y has two components Y_1 and Y_2. The proof for the case that Y is connected is similar. The ribbon graph cobordism (W_B, Γ_B) is shown in the left subfigure of Figure A.9 and suppose endpoints of Γ_B correspond to w'_1 and w'_2 in $\widetilde{Y}$. The proof is essentially the same as that in Step 2. We first change the position of W_B and add two arcs to Γ_B to obtain (W'_B, Γ'_B), as shown in the middle subfigure of Figure A.9. Second, we choose Y_B in the neck of W'_B and set G_B to be the images of Σ in $\partial W'_B$ and $\partial N(Y_B)$. Third, we replace $N(Y_B)$ by $\Sigma \times D^2 \sqcup (-\Sigma \times D^2)$ via Lemma A.1.35 to obtain (W''_B, Γ''_B), as shown in the right subfigure of Figure A.9. Finally we remove arcs in the interior of the cobordism and show it is the identity map because

$$(W''_B, \Gamma''_B) \cong (\widetilde{Y} \times I, (w'_1 \sqcup w'_2) \times I).$$

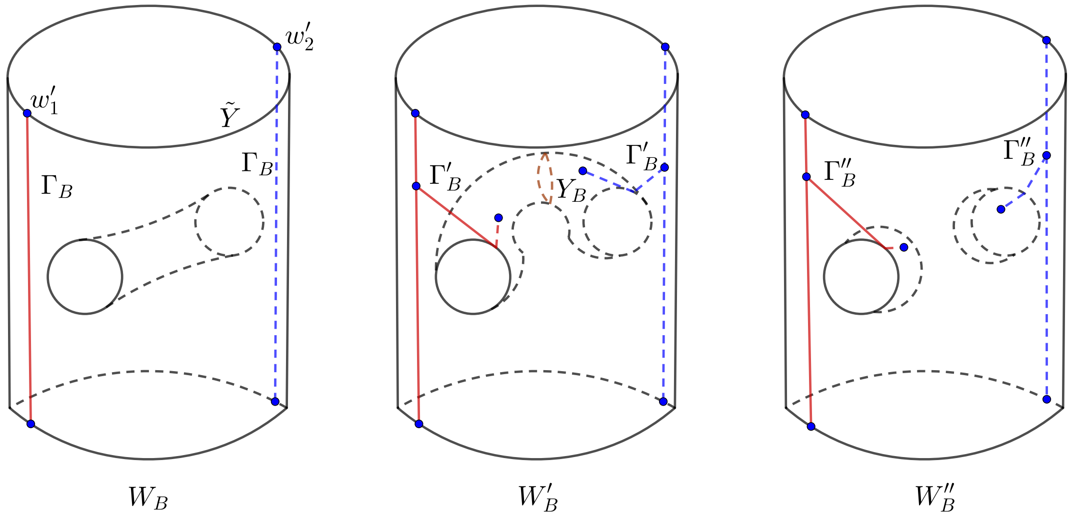

Figure A.9 Ribbon graph cobordisms $(W_B, \Gamma_B), (W'_B, \Gamma'_B)$, and (W''_B, Γ''_B).

Finally, we know Step 2 and Step 3 imply

$$HF(Y|F) \cong HF(\widetilde{Y}|\widetilde{F})$$

via cobordism maps associated to ribbon graph cobordisms

$$(W, \Gamma_A \cap W) \cong (W, \Gamma_B \cap W) \text{ and } (\bar{W}, \Gamma_A \cap \bar{W}) \cong (\bar{W}, \Gamma_B \cap \bar{W}).$$

Note that those ribbon graph cobordisms are restricted in the sense of Definition A.1.2.

□

A.2 Sutured Heegaard Floer homology

A.2.1 Two equivalent constructions

In this subsection, we introduce two equivalent definitions of sutured Heegaard Floer homology. The first one is due to Juhász [Juh06], based on balanced diagrams of balanced sutured manifolds. The other follows from the construction by Kronheimer-Mrowka [KM10b] and Baldwin-Sivek [BS15], based on Floer's excision theorem in Subsection A.1.3. These definitions are denoted by SFH and **SHF**, respectively. The equivalence of these definitions was shown by Lekili [Lek13] and Baldwin-Sivek [BS21c]. We will focus on the equality for graded Euler characteristics of those two constructions in Subsection A.2.3.

Definition A.2.1 ([Juh06, Section 2]). A **balanced diagram** $\mathcal{H} = (\Sigma, \alpha, \beta)$ is a tuple satisfying the following.

(1) Σ is a compact, oriented surface with boundary.

(2) $\alpha = \{\alpha_1, \dots, \alpha_n\}$ and $\beta = \{\beta_1, \dots, \beta_n\}$ are two sets of pairwise disjoint simple closed curves in the interior of Σ.

(3) The maps $\pi_0(\partial\Sigma) \to \pi_0(\Sigma\backslash\alpha)$ and $\pi_0(\partial\Sigma) \to \pi_0(\Sigma\backslash\beta)$ are surjective.

For such triple, let N be the 3-manifold obtained from $\Sigma \times [-1,1]$ by attaching 3–dimensional 2–handles along $\alpha_i \times \{-1\}$ and $\beta_i \times \{1\}$ for $i = 1, \dots, n$ and let $\nu = \partial\Sigma \times \{0\}$. A balanced diagram (Σ, α, β) is called **compatible** with a balanced sutured manifold (M, γ) if the balanced sutured manifold (N, ν) is diffeomorphic to (M, γ).

Suppose $\mathcal{H} = (\Sigma, \alpha, \beta)$ is a balanced diagram with $g = g(\Sigma)$ and $n = |\alpha| = |\beta|$. Suppose $\mathcal{H}$ satisfies the admissible condition in [Juh06, Section 3]. Consider two tori

$$\mathbb{T}_\alpha := \alpha_1 \times \cdots \times \alpha_n \text{ and } \mathbb{T}_\beta := \beta_1 \times \cdots \times \beta_n$$

in the symmetric product

$$\mathrm{Sym}^n\Sigma := (\prod_{i=1}^n \Sigma)/S_n.$$

The chain complex $SFC(\mathcal{H})$ is a free $\mathbb{F}_2$-module generated by intersection points $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Similar to the construction of CF^-, for a generic path of almost complex structures J_s on $\mathrm{Sym}^n\Sigma$, define the differential on $SFC(\mathcal{H})$ by

$$\partial_{J_s}(\boldsymbol{x}) = \sum_{\boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}) \\ \mu(\phi) = 1}} \#\widehat{\mathcal{M}}_{J_s}(\phi) \cdot \boldsymbol{y}.$$

Theorem A.2.2 ([Juh06, JTZ21]). *Suppose (M, γ) is a balanced sutured manifold. Then there is an admissible balanced diagram $\mathcal{H}$ compatible with (M, γ). The vector spaces $H(SFC(\mathcal{H}), \partial_{J_s})$ for different choices of $\mathcal{H}$ and J_s, together with some canonical maps, form a transitive system over $\mathbb{F}_2$. Let $SFH(M, \gamma)$ denote this transitive system and also the* *associated actual group. Moreover, there is a decomposition*

$$SFH(M,\gamma) = \bigoplus_{\mathfrak{s}\in \mathrm{Spin}^c(M,\partial M)} SFH(M,\gamma,\mathfrak{s}).$$

Then we define the second version of sutured Heegaard Floer homology.

Definition A.2.3. Suppose (M,γ) is a balanced sutured manifold and (Y,R) is a closure of (M,γ) as in Theorem 2.3.10 (we omit ω). Define

$$SHF(M,\gamma) := HF(Y|R) = \bigoplus_{\mathfrak{s}\in \mathrm{Spin}^c(Y|R)} HF^+(Y,\mathfrak{s}).$$

Remark A.2.4. By work of Kutluhan-Lee-Taubes [KLT20], for any $\mathfrak{s}\in \mathrm{Spin}^c(Y)$, there is an isomorphism

$$HF^+(Y,\mathfrak{s}) \cong \widecheck{HM}_*(Y,\mathfrak{s}) = \widecheck{HM}_\bullet(Y,\mathfrak{s}).$$

The last group is used to define SHM in [KM10b].

Based on Floer's excision theorem and the construction in [BS15] (see also [LY21b, Section 2]), we can prove the naturality of $SHF(M,\gamma)$. Let $\mathbf{SHF}(M,\gamma)$ be the projectively transitive system which is the untwisted refinement of $SHF(M,\gamma)$. *A priori*, it depends on the choice of a large genus $g(R)$ of the closure (Y,R). But we omit the choice in the notation.

A.2.2 Gradings associated to admissible surfaces

In this subsection, we discuss the gradings on SFH associated to admissible surfaces.

For a balanced sutured manifold (M,γ), we can decompose $SFH(M,\gamma)$ along spin^c structures.

Fix a Riemannian metric g on M. Let v_0 be a nowhere vanishing vector field along ∂M that points into M along $R_-(\gamma)$, points out of M along $R_+(\gamma)$, and on γ it is the gradient of the height function $A(\gamma)\times I\to I$. The space of such vector fields is contractible, so the choice of v_0 is not important.

Suppose v and w are nowhere vanishing vector fields on M that agree with v_0 on ∂M. They are called **homologous** if there is an open ball $B\subset \mathrm{int} M$ such that v and w are homotopic on $M\backslash B$ through nowhere vanishing vector fields rel ∂M. Let $\mathrm{Spin}^c(M,\gamma)$ be the set of homology classes of nowhere vanishing vector fields v on M with $v|_{\partial M}=v_0$. Note that $\mathrm{Spin}^c(M,\gamma)$ is an affine space over $H^2(M,\partial M)$.

Suppose $\mathcal{H} = (\Sigma, \alpha, \beta)$ is a balanced diagram compatible with (M, γ). For each intersection point $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we can assign a spin^c structure $\mathfrak{s}(\boldsymbol{x}) \in \mathrm{Spin}^c(M, \gamma)$ as follows (*c.f.* [Juh06, Section 4]).

we choose a self-indexing Morse function $f : M \to [-1, 4]$ such that

$$f^{-1}(\frac{3}{2}) = \Sigma \times \{0\}.$$

Moreover, curves α, β are intersections of $\Sigma \times \{0\}$ with the ascending and descending manifolds of the index 1 and 2 critical points of f, respectively. Then any intersection point of $\alpha_i \subset \alpha$ and $\beta_j \subset \beta$ corresponds to a trajectory of grad$f$ connecting a index 1 critical point to a index 2 critical point. For $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, let $\gamma_{\boldsymbol{x}}$ be the multi-trajectory corresponding to intersection points in $\boldsymbol{x}$.

In a neighborhood $N(\gamma_{\boldsymbol{x}})$, we can modify grad$f$ to obtain a nowhere vanishing vector field v on M such that $v|_{\partial M} = v_0$. Let $\mathfrak{s}(\boldsymbol{x}) \in \mathrm{Spin}^c(M, \gamma)$ be the homology class of this vector field v.

From the assignment of the spin^c structure, we have the following proposition.

Proposition A.2.5. *For any* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$*, we have*

$$\mathfrak{s}(\boldsymbol{x}) - \mathfrak{s}(\boldsymbol{y}) = \mathrm{PD}([\gamma_{\boldsymbol{x}} - \gamma_{\boldsymbol{y}}]),$$

where $\mathrm{PD} : H_1(M) \to H^2(M, \partial M)$ *is the Poincaré duality map.*

It can be shown that there is no differential between generators corresponding to different spin^c structures. Hence we have the following decomposition.

Proposition A.2.6 ([Juh06])**.** *For any balanced sutured manifold* (M, γ)*, there is a decomposition*

$$SFH(M, \gamma) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(M, \partial M)} SFH(M, \gamma, \mathfrak{s}).$$

Suppose $S \subset (M, \gamma)$ is an admissible surface S. To associate a $\mathbb{Z}$-grading on $SFH(M, \gamma)$ similar to Subsection 2.3.3, we need to suppose (M, γ) is **strongly balanced**, *i.e.* for every component F of ∂M, we have

$$\chi(F \cap R_+(\gamma)) = \chi(F \cap R_-(\gamma)).$$

Remark A.2.7. If ∂M is connected, then it is automatically strongly balanced. For any balanced sutured manifold (M, γ), we can obtain a strongly balanced manifold (M', γ') by attaching contact 1-handles [Juh08, Remark 3.6]. In Subsection A.2.5, we will show

$$SFH(M',\gamma') \cong SFH(M,\gamma)$$

and this isomorphism respects spin^c structures. Hence we can always deal with a strongly balanced manifold without losing any information.

Convention. When discussing the $\mathbb{Z}$-grading on $SFH(M,\gamma)$ associated to an admissible surface $S \subset (M,\gamma)$, we always suppose (M,γ) is strongly balanced.

The following construction is based on [Juh08, Section 3].

Let $v_0^\perp$ be the plane bundle perpendicular to v_0 under the fixing Riemannian metric g. The strongly balanced condition on (M,γ) ensures that $v_0^\perp$ is trivial (*c.f.* [Juh08, Proposition 3.4]). Let t be a trivialization of $v_0^\perp$. Since any spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma)$ can be represented by a nonvanishing vector field v on M with $v|_{\partial M} = v_0$, we can define the relative Chern class

$$c_1(\mathfrak{s},t) := c_1(v^\perp,t) \in H^2(M,\partial M)$$

by considering the plane bundle $v^\perp$ perpendicular to v.

Let v_S be the positive unit normal field of S. For a generic S, we can suppose v_S is nowhere parallel to v_0 along ∂S. Let $p(v_S)$ be the projection of v_S into $v_0^\perp$. Note that $p(v_S)|_{\partial S}$ is nowhere zero. Suppose the components of ∂S are $T_1,\ldots,T_k$, oriented by the boundary orientation.

For $i = 1,\ldots,k$, Let $r(T_i,t)$ be the rotation number $p(v_S)|_{T_i}$ with respect to the trivialization t as we go around T_i. Moreover, define

$$r(S,t) := \sum_{i=1}^{k} r(T_i,t).$$

Suppose $T_1,\ldots,T_k$ intersect γ transversely. Define

$$c(S,t) = \chi(S) - \frac{1}{2}|\partial S \cap \gamma| - r(S,t). \tag{A.2.1}$$

Remark A.2.8. The original definition of $c(S,t)$ in [Juh08, Section 3] involves the index $I(S)$, which is equal to $\frac{1}{2}|\partial S \cap \gamma|$ when $T_1,\ldots,T_k$ intersect γ transversely (*c.f.* [Juh08, Lemma 3.9]).

Suppose t_S is the trivialization of $v_0^{\perp}$ induced by $p(v_S)|_{\partial S}$. Then for any $v^{\perp}$ with $v^{\perp}|_{\partial M} = v_0^{\perp}$ and any trivialization t of $v_0^{\perp}$, we have

$$\langle c_1(v^{\perp}, t_S) - c_1(v^{\perp}, t), [S]\rangle = r(S,t) \tag{A.2.2}$$

(*c.f.* Proof of [Juh08, Lemma 3.10]; see also [Juh10, Lemma 3.11]).

Definition A.2.9. Consider the construction as above. Define

$$SFH(M,\gamma,S,i) := \bigoplus_{\substack{\mathfrak{s}\in \mathrm{Spin}^c(M,\gamma)\\ \langle c_1(\mathfrak{s},t_S),[S]\rangle = -2i}} SFH(M,\gamma,\mathfrak{s}). \tag{A.2.3}$$

Remark A.2.10. The minus sign of $(2i)$ is to make this definition parallel to the $\mathbb{Z}$-grading on $\underline{\mathrm{SHI}}(M,\gamma)$ associated to S. See the proofs of the following propositions.

Proposition A.2.11. *The decomposition in Definition A.2.9 satisfies Terms (1)-(5) in Theorem 2.3.20, replacing* $\underline{\mathrm{SHI}}$ *by* SFH.

Proof. Term (1) follows from the adjunction inequality in [Juh10, Theorem 2]. Note that if $2i = |\partial S \cap \gamma| - \chi(S)$, then for $\mathfrak{s}$ corresponds to $SFH(M,\gamma,S,i)$, we have

$$\langle c_1(\mathfrak{s},t_S),[S]\rangle = \chi(S) - |\partial S\cap\gamma| = c(S,t_S), \tag{A.2.4}$$

where the last equality follows from (A.2.1) and (A.2.2).

Term (2) follows from [Juh08, Lemma 3.10] and (A.2.4).

Terms (3)-(5) follow from definitions and symmetry on balanced diagrams. □

Proposition A.2.12. *Consider the stabilized surfaces S^p and S^{p+2k} in Theorem 2.3.28. Then for any $l \in \mathbb{Z}$, we have*

$$SFH(M,\gamma,S^p,l) = SFH(M,\gamma,S^{p+2k},l+k).$$

Proof. Suppose S^+ and S^- are positive and negative stabilizations of S. Since the stabilization operation is local, we have the following equation by direct calculation

$$r(S^+,t) = r(S,t) - 1$$

for any trivialization t of $v_0^{\perp}$. Note that $[S^+]=[S]$. Hence for $\mathfrak{s}\in\mathrm{Spin}^c(M,\gamma)$ corresponds to $SFH(M,\gamma,S,i)$, we have

$$\begin{aligned}\langle c_1(\mathfrak{s},t_{S^+}),[S^+]\rangle =&\langle c_1(\mathfrak{s},t_S),[S^+]\rangle + r(S^+,t_S)\\ =&\langle c_1(\mathfrak{s},t_S),[S^+]\rangle + r(S,t_S)-1\\ =&\langle c_1(\mathfrak{s},t_S),[S^+]\rangle -1\\ =&\langle c_1(\mathfrak{s},t_S),[S]\rangle -1\\ =&-2i-1.\end{aligned}$$

Applying this calculation for $(2k)$ times gives the desired result. □

Proposition A.2.13. *Suppose S_1 and S_2 are two admissible surfaces in (M,γ) such that*

$$[S_1]=[S_2]=\alpha\in H_2(M,\partial M).$$

Then there exists a constant C so that

$$\underline{\mathrm{SHI}}(M,\gamma,S_1,l)=\underline{\mathrm{SHI}}(M,\gamma,S_2,l+C).$$

Proof. This follows directly from the definition. □

A.2.3 Euler characteristics

Definition A.2.14. For a balanced sutured manifold (M,γ), let the $\mathbb{Z}_2$-grading of $SFH(M,\gamma)$ be induced by the sign of intersection points of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ for some compatible diagram $\mathcal{H}=(\Sigma,\alpha,\beta)$ (*c.f.* [FJR09, Section 3.4]). Suppose $H=H_1(M),H'=H_1(M)/\mathrm{Tors}$ and suppose $p:H\to H'$ is the projection map. Recall the definition of $\chi(SFH(M,\gamma))$ in (1.2.2): Fixing a spin^c structure $\mathfrak{s}_0$, define

$$\chi(SFH(M,\gamma)):=\sum_{\mathfrak{s}\in\mathrm{Spin}^c(M,\gamma)}\chi(SFH(M,\gamma,\mathfrak{s}))\cdot\mathrm{PD}(\mathfrak{s}-\mathfrak{s}_0))\in\mathbb{Z}[H]/\pm H,$$

where PD is the Poincaré duality map. Let

$$\chi_{\mathrm{gr}}(SFH(M,\gamma))\in\mathbb{Z}[H']/\pm H'$$

be the induced element from $\chi(SFH(M,\gamma))$ under the projection $p:H\to H'$.

Based on the gradings associated to admissible surfaces, define

$$\chi_{\mathrm{gr}}(\mathbf{SHF}(M,\gamma)) \in \mathbb{Z}[H']/\pm H'$$

similarly to $\chi_{\mathrm{gr}}(\underline{\mathrm{SHI}}(M,\gamma))$ in Definition 2.3.30. Note that the $\mathbb{Z}_2$-grading is also from the sign of intersection points of two tori in the symmetric product defining HF^- of the closure of (M,γ).

Theorem A.2.15 ([Lek13, Theorem 24], see also [BS21c, Theorem 3.26])**.** *Suppose (M,γ) is a balanced sutured manifold and (Y,R) is a closure of (M,γ). Then there exists a balanced diagram $\mathcal{H}=(\Sigma,\alpha,\beta)$ compatible with (M,γ) and a singly-pointed Heegaard diagram $\mathcal{H}'=(\Sigma',\alpha',\beta',z)$ of Y so that the following holds.*

(1) Σ is a submanifold of Σ'.

(2) α and β are subsets of α' and β', respectively.

(3) Suppose $\alpha'=\alpha\cup\alpha''$ and $\beta'=\beta\cup\beta''$. There exists an intersection point $\boldsymbol{x}_1\in\mathbb{T}_{\alpha''}\cap\mathbb{T}_{\beta''}$ so that the map

$$\begin{aligned} f: SFC(\mathcal{H}) &\to CF^+(\mathcal{H}'|R)\\ \boldsymbol{c} &\mapsto \boldsymbol{c}\times\boldsymbol{x}_1\end{aligned}$$

is a quasi-isomorphism, where $CF^+(\mathcal{H}'|R)$ is the chain complex of $HF^+(Y|R)$ associated to $\mathcal{H}'$.

Corollary A.2.16. *Suppose (M,γ) is a balanced sutured manifold and $H'=H_1(M)/\mathrm{Tors}$. We have*

$$SFH(M,\gamma)\cong\mathbf{SHF}(M,\gamma)$$

with respect to the grading associated to H and the $\mathbb{Z}_2$-grading, up to a global grading shift.

In particular, we have

$$\chi_{\mathrm{gr}}(SFH(M,\gamma))=\chi_{\mathrm{gr}}(\mathbf{SHF}(M,\gamma))\in\mathbb{Z}[H]/\pm H,$$

where $\chi_{\mathrm{gr}}(\mathbf{SHF}(M,\gamma))$ is defined as in Definition 2.3.30.

Proof. It suffices to show the quasi-isomorphism in Theorem A.2.15 respects spin^c structures and $\mathbb{Z}_2$-gradings.

Consider the $\mathbb{Z}_2$-gradings at first. Suppose $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are two generators of $SFC(\mathcal{H})$. Note that the $\mathbb{Z}_2$-grading of $\boldsymbol{c}_i$ is defined by the sign of the corresponding intersection point in

$\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ for $i = 1,2$. For $\boldsymbol{c}_i \times \boldsymbol{x}_1$, the $\mathbb{Z}_2$-grading is defined by mod 2 Maslov grading, which coincides with the sign of the corresponding intersection point in $\mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$. Thus, we have

$$\mathrm{gr}_2(\boldsymbol{c}_1) - \mathrm{gr}_2(\boldsymbol{c}_2) = \mathrm{gr}_2(\boldsymbol{c}_1 \times \boldsymbol{x}_1) - \mathrm{gr}_2(\boldsymbol{c}_2 \times \boldsymbol{x}_1),$$

where gr_2 is the $\mathbb{Z}_2$-grading.

Then we consider spin^c structures. Consider $\boldsymbol{c}_i$ for $i = 1,2$ as above. From [Juh06, Lemma 4.7], there is a one chain $\gamma_{\boldsymbol{c}_1} - \gamma_{\boldsymbol{c}_2}$ such that

$$\mathfrak{s}(\boldsymbol{c}_1) - \mathfrak{s}(\boldsymbol{c}_2) = \mathrm{PD}([\gamma_{\boldsymbol{c}_1} - \gamma_{\boldsymbol{c}_2}]),$$

where $\mathfrak{s}(\cdot) : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathrm{Spin}^c(M, \partial M)$ is defined in [Juh06, Definition 4.5], and $\mathrm{PD} : H_1(M) \to H^2(M, \partial M)$ is the Poincaré duality map.

From [OS04d, Lemma 2.19], we have

$$\mathfrak{s}_z(\boldsymbol{c}_1 \times \boldsymbol{x}_1) - \mathfrak{s}_z(\boldsymbol{c}_2 \times \boldsymbol{x}_1) = \mathrm{PD}'(i_*([\gamma_{\boldsymbol{c}_1} - \gamma_{\boldsymbol{c}_2}])),$$

where $\mathfrak{s}_z(\cdot) : \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'} \to \mathrm{Spin}^c(Y)$ is defined in [OS04d, Section 2.6] and $\mathrm{PD}' : H_1(Y) \to H^2(Y)$ is the Poincaré duality map, and $i_* : H_1(M) \to H_1(Y)$ is the map induced by inclusion $i : M \to Y$.

Hence we have

$$c_1(\mathfrak{s}_z(\boldsymbol{c}_1 \times \boldsymbol{x}_1)) - c_1(\mathfrak{s}_z(\boldsymbol{c}_2 \times \boldsymbol{x}_1)) = 2\mathrm{PD}'(i_*([\gamma_{\boldsymbol{c}_1} - \gamma_{\boldsymbol{c}_2}])).$$

Finally, the argument about graded Euler characteristics follows from definitions. □

A.2.4 Surgery exact triangle

Suppose (M, γ) is a balanced sutured manifold and K is a knot in M. Consider three balanced sutured manifolds (M_i, γ_i) for $i = 1,2,3$ obtained from (M, γ) by Dehn surgeries along K. If the Dehn filling curves $\eta_1, \eta_2, \eta_3 \subset \partial(M \backslash \mathrm{int} \partial N(K))$ satisfy

$$\eta_1 \cdot \eta_2 = \eta_2 \cdot \eta_3 = \eta_3 \cdot \eta_1 = -1,$$

then we have the following exact triangle for sutured instanton homology from the surgery exact triangle (2.3.3) in the closure of (M_i, γ_i)

$$\begin{array}{ccc} \underline{\mathrm{SHI}}(M_1,\gamma_1) & \longrightarrow & \underline{\mathrm{SHI}}(M_2,\gamma_2) \\ & \nwarrow \quad \swarrow & \\ & \underline{\mathrm{SHI}}(M_3,\gamma_3) & \end{array} \tag{A.2.5}$$

In this subsection, we show the exact triangle (A.2.5) is also true when replacing $\underline{\mathrm{SHI}}$ by SFH.

First, we quickly review Juhász's construction of the cobordism map associated to a Dehn surgery (*c.f.* [Juh16, Section 6], see also [OS06a] for Dehn surgeries on closed 3-manifolds).

For simplicity, suppose η_1 is the meridian of K. Choose an arc a connecting K to $R_+(\gamma)$. We can construct a sutured triple diagram $(\Sigma, \alpha, \beta, \delta)$ satisfying the following properties.

1. $|\alpha| = |\beta| = |\gamma| = d$.
2. $(\Sigma, \alpha, \{\beta_2, \ldots, \beta_d\})$ is a diagram of $(M', \gamma') = (M \backslash N(K \cup a), \gamma)$.
3. $\delta_2, \ldots, \delta_d$ are obtained from $\beta_2, \ldots, \beta_d$ by small isotopy, respectively.
4. After compressing Σ along $\beta_2, \ldots, \beta_d$, the induced curves β_1 and δ_1 lie in the punctured torus $\partial N(K) \backslash N(a)$.
5. β_1 represents the meridian η_1 of K and δ_1 represents the curve η_2. In particular, β_1 intersects δ_1 transversely at one point.

Then we can construct a 4-manifold $\mathcal{W}_{\alpha,\beta,\delta}$ associated to $(\Sigma, \alpha, \beta, \delta)$ such that it is a cobordism from $(M, \gamma) = (M_1, \gamma_1)$ to

$$(M_2, \gamma_2) \sqcup (R_+ \times I \times \partial R_+ \times I) \#^{d-n} (S^2 \times S^1),$$

where $R_+ = R_+(\gamma)$ and different copies of $S^2 \times S^1$ might be summed along different components of $R_+ \times I$.

Choose a top dimensional generator $\Theta_{\beta,\delta}$ of

$$SFH(R_+ \times I \times \partial R_+ \times I) \#^{d-n} (S^2 \times S^1) \cong \Lambda^* H^1(\#^{d-n}(S^2 \times S^1)).$$

Note that (Σ, α, β) is a balanced diagram of (M_1, γ_1) and (Σ, α, δ) is a balanced diagram of (M_2, γ_2). There is a map

$$F_{\alpha,\beta,\gamma} : SFH(\Sigma, \alpha, \beta) \otimes SFH(\Sigma, \beta, \delta) \to SFH(\Sigma, \alpha, \delta)$$

obtained by counting holomorphic triangles in $(\Sigma,\alpha,\beta,\delta)$. Then define the cobordism map as

$$\begin{aligned} F_1 : SFH(M_1,\gamma_1) &\to SFH(M_2,\gamma_2) \\ F_1(x) &= F_{\alpha,\beta,\delta}(x,\Theta_{\beta,\delta}) \end{aligned}$$

Similarly, we can define the cobordism maps F_2 and F_3.

Theorem A.2.17 (Surgery exact triangle)**.** *Consider* (M_i,γ_i) *and cobordism maps* F_i *for* $i = 1,2,3$ *as above. Then we have an exact triangle*

$$\begin{array}{ccc} SFH(M_1,\gamma_1) & \xrightarrow{F_1} & SFH(M_2,\gamma_2) \\ {\scriptstyle F_3}\nwarrow & & \swarrow{\scriptstyle F_2} \\ & SFH(M_3,\gamma_3) & \end{array} \tag{A.2.6}$$

Proof. The proof follows the proof of [OS04c, Theorem 9.12] without essential changes (see also [OS05c, OS06b]). Since the cobordism maps F_i are well-defined on SFH, we can verify the exact triangle for any diagram. We can construct a diagram $(\Sigma,\alpha,\beta,\delta,\zeta)$ such that $(\Sigma,\alpha,\beta,\delta)$ defines F_1, $(\Sigma,\alpha,\delta,\zeta)$ defines F_2, and $(\Sigma,\alpha,\zeta,\beta)$ defines F_3. Then we can verify the assumptions of the triangle detection lemma [OS05c, Lemma 4.2] by counting holomorphic squares and pentagons and then this lemma induces the desired exact triangle. □

A.2.5 Contact handles and bypasses

Suppose $(M,\gamma) \subset (M',\gamma')$ is a proper inclusion of balanced sutured manifolds and suppose ξ is a contact structure on $M'\backslash\text{int}M$ with dividing sets $\gamma' \cup (-\gamma)$. Honda-Kazez-Matić [HKM08] defined a map

$$\Phi_\xi : SFH(M,\gamma) \to SFH(M',\gamma'),$$

which is indeed the motivation of Baldwin-Sivek's construction in Subsection 2.3.4.

Originally, this map is defined by partial open book decompositions, and there are some technical conditions. Juhász-Zemke [JZ20] provided an alternative description of this map by contact handle decompositions. Their description is explicit on balanced diagrams of sutured manifolds. We will follow this alternative definition and describe the maps for contact 1- and 2-handle attachments.

It is also worth mentioning that Zarev [Zar10] defined a gluing operation for sutured manifolds and conjectured the map associated to contact structures above can be recovered by the gluing operation. This was proved by Leigon and Salmoiraghi [LS20].

Juhász-Zemke's construction can be shown in Figure A.10 and Figure A.11 ([JZ20, Figure 1.1]). Note that for all maps associated to contact structures, we should reverse the orientations of the manifold and the suture.

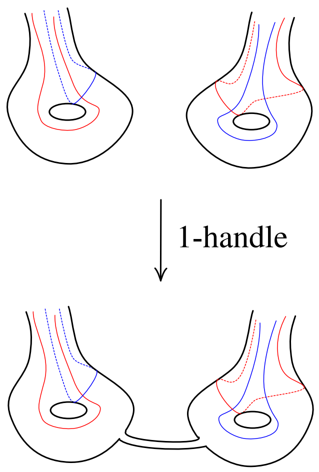

Figure A.10 Contact 1-handle.

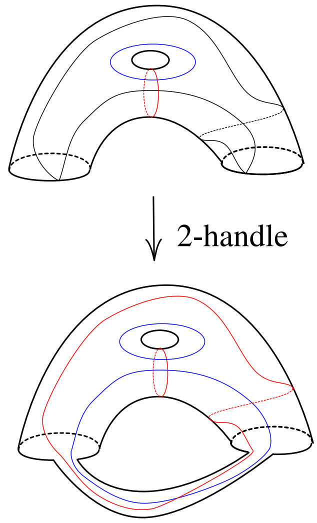

Figure A.11 Contact 2-handle.

Let (Σ,α,β) be a balanced diagram compatible with (M,γ). Then $(-\Sigma,\alpha,\beta)$ is a balanced diagram compatible with $(-M,-\gamma)$. Attaching a (3-dimensional) contact 1-handle along D_+ and D_- corresponds to attaching a 2-dimensional 1-handle along $D_+\cap\gamma$ and $D_-\cap\gamma$ in $\partial\Sigma$. This operation does not change the sutured Floer chain complex and we define $C_{h^1}=C_{h^1,D_+,D_-}$ as the tautological map on intersection points.

For a contact 2-handle attachment along $\mu\subset\partial M$, note that $|\mu\cap\gamma|=2$. Suppose λ_+ and λ_- are arcs corresponding to $\mu\cap R_+(\gamma)$ and $\mu\cap R_-(\gamma)$, respectively. After isotopy, we can suppose λ_+ and λ_- are propertly embedded arcs on Σ. We glue a 2-dimensional 1-handle h along $\partial\Sigma$ to obtain Σ', and construct two curves α_0 and β_0 that intersect at one point c in H, and such that

$$\alpha_0\cap\Sigma=\lambda_+,\beta_0\cap\Sigma=\lambda_-.$$

Consider the balanced diagram $(\Sigma',\alpha\cup\{\alpha_0\},\beta\cup\{\beta_0\})$ and define the map associated to the contact 2-handle attachment as

$$C_{h^2}(\boldsymbol{x})=C_{h^2,\mu}(\boldsymbol{x}):=\boldsymbol{x}\times\{c\}$$

for any $\boldsymbol{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$.

Since a bypass attachment can be regarded as a composition of a contact 1-handle and 2-handle attachment (*c.f.* Subsection 2.3.4), we can define the bypass map by $C_{h^2}\circ C_{h^1}$.

Honda [Hon] proposed an exact triangle associated to bypass maps for SFH, which is indeed the motivation of the bypass exact triangle in Theorem 2.3.38. A proof of the exact triangle based on bordered sutured Floer homology was provided by Etnyre-Vela-Vick-Zarev [EVVZ17].

Theorem A.2.18 (Bypass exact triangle, [EVVZ17, Section 6])**.** *Suppose (M,γ_1), (M,γ_2), (M,γ_3) are balanced sutured manifolds such that the underlying 3-manifolds are the same, and the sutures γ_1, γ_2, and γ_3 only differ in a disk shown in Figure 2.5. Then there exists an exact triangle*

$$\begin{array}{ccc} SFH(-M,-\gamma_1) & \xrightarrow{\psi_1} & SFH(-M,-\gamma_2) \\ {\scriptstyle\psi_3}\nwarrow & & \swarrow{\scriptstyle\psi_2} \\ & SFH(-M,-\gamma_3) & \end{array}$$

where ψ_1,ψ_2,ψ_3 are bypass maps associated to the corresponding bypass arcs.

From Juhász-Zemke's description of contact gluing maps, it is obvious that the maps respect the decomposition of SFH by spin^c structures. We describe this fact explicitly as follows.

Lemma A.2.19. *Suppose (M,γ) is a balanced sutured manifold and suppose (M',γ') is the resulting sutured manifold after either a contact 1-handle or 2-handle attachment. For any spin^c structure $\mathfrak{s}\in \mathrm{Spin}^c(-M,-\gamma)$, suppose $\mathfrak{s}'\in \mathrm{Spin}^c(-M',-\gamma')$ is its extension corresponding to handle attachments. Then we have*

$$C_{h^i}(SFH(-M,-\gamma,\mathfrak{s}))\subset SFH(-M',-\gamma',\mathfrak{s}'),$$

where $i\in\{1,2\}$.

Proof. We prove the claim on the chain level. After fixing a spin^c structure $\mathfrak{s}_0$ on (M,γ), we can identify $\mathrm{Spin}^c(M,\gamma)$ with $H^2(M,\partial M)\cong H_1(M)$. Moreover, we can represent the difference of two spin^c structures by a one-cycle in Proposition A.2.5.

We can extend $\mathfrak{s}_0$ to a spin^c structure $\mathfrak{s}_0'$ on (M,γ) and identify $\mathrm{Spin}^c(M',\gamma')$ with $H_1(M')$. The inclusion $i: M\to M'$ induces a map

$$i_*: H_1(M)\to H_1(M').$$

For any $\boldsymbol{x},\boldsymbol{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$, the one cycle $\gamma_{\boldsymbol{x}}-\gamma_{\boldsymbol{y}}$ defined in Proposition A.2.5 lies in the interior of M.

For a contact 1-handle, since the associated map C_{h^1} is tautological on intersection points, the homology class $i_*([\gamma_{\boldsymbol{x}} - \gamma_{\boldsymbol{y}}])$ characterizes the difference of spin^c structures on (M', γ') for $\boldsymbol{x}$ and $\boldsymbol{y}$.

For a contact 2-handle, since $\gamma_{\boldsymbol{x}\times\{c\}}$ is the union of multi-trajectory $\gamma_{\boldsymbol{x}}$ and the trajectory associated to c, we have

$$[\gamma_{\boldsymbol{x}\times\{c\}} - \gamma_{\boldsymbol{y}\times\{c\}}] = i_*([\gamma_{\boldsymbol{x}} - \gamma_{\boldsymbol{y}}]).$$

This implies the desired proposition. □

Remark A.2.20. The reader can compare Lemma A.2.19 with Proposition 4.1.6. Note that when $H_1(M)$ has torsions, preserving the spin^c structures is stronger than preserving the gradings associated to an admissible surface.

Corollary A.2.21. *Suppose α is a bypass arc on a balanced sutured manifold (M, γ). Suppose (M, γ') is the resulting manifold after the bypass attachment along α. Then the bypass map ψ_α for SFH respects spin^c structures, i.e., for any $\mathfrak{s} \in \mathrm{Spin}^c(M, \gamma)$ and its extension $\mathfrak{s}' \in \mathrm{Spin}^c(M, \gamma')$, we have*

$$\psi_\alpha(SFH(-M, -\gamma, \mathfrak{s})) \subset SFH(-M, -\gamma', \mathfrak{s}').$$

Proof. This follows directly from Lemma A.2.19 by the fact that a bypass attachment is a composition of a contact 1-handle and 2-handle attachment. □

Remark A.2.22. By Corollary A.2.21, if we consider the $\mathbb{Z}$-grading associated to an admissible surface S in Subsection A.2.9, then the bypass exact triangle in Theorem A.2.18 satisfies the similar grading shifting property to that in Lemma 3.1.6.

For sutured instanton homology, the map associated to a contact 2-handle is defined by the composition of the inverse of a contact 1-handle map and the cobordism map of a 0-surgery. The following proposition shows that we can define the map C_{h^2} for SFH in the same way.

Lemma A.2.23 ([GZ])**.** *Suppose (M, γ) is a balanced sutured manifold and (M', γ') is the resulting sutured manifold after a contact 2-handle attachment along $\mu \subset \partial M$. Let μ' be the framed knot obtained by pushing μ into the interior of M slightly, with the framing induced from ∂M. Suppose (N, γ_N) is the sutured manifold obtained from (M, γ) by a 0-surgery along μ'. Let*

$$F_{\mu'} : SFH(-M, -\gamma) \to SFH(-N, -\gamma_N)$$

be the associated map. Let $D \subset N$ be the product disk which is the union of the annulus bounded by $\mu \cup \mu'$ and the meridian disk of the filling solid torus. Let

$$C_D : SFH(-N, -\gamma_N) \to SFH(-M', -\gamma')$$

be the map associated to the decomposition along D (i.e. the inverse of a contact 1-handle map). Then we have

$$C_{h^2,\mu} = C_D \circ F_{\mu'} : SFH(-M, -\gamma) \to SFH(-M', -\gamma').$$

Proof. Since all maps are well-defined on SFH, we can verify the claim by any diagram. Suppose (Σ, α, β) is a balanced diagram compatible with (M, γ). We note that the map associated to the 0-surgery along μ' may be achieved by first performing a compound stabilization and then computing a triangle map. The resulting diagram leaves an extra band which is deleted by C_D. By [OS04d, Theorem 9.4], the claim then follows from a model computation in the stabilization region, as shown in Figure A.12. □

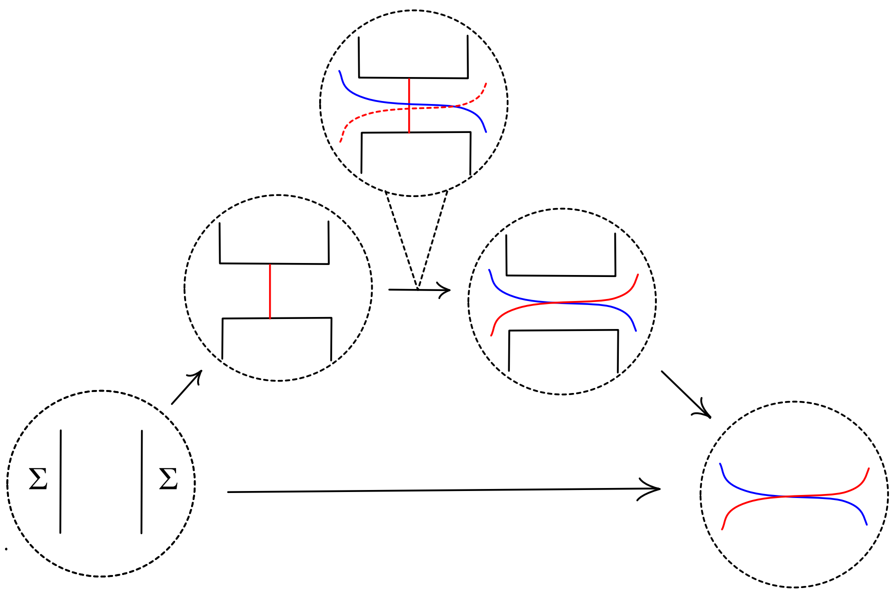

Figure A.12 Realizing the contact 2-handle map (bottom-most long arrow) as a composition of a compound stabilization (top), followed by a 4-dimensional 2-handle map (middle left), followed by a product disk map (middle right). A holomorphic triangle of the 2-handle map is indicated in the top subfigure.

Combining the surgery exact triangle in Theorem A.2.17 with Lemma A.2.23, we obtain similar results in Lemma 3.1.8 for SFH.

Proposition A.2.24. *Consider the setups in Subsection 3.1.1. Suppose $T' = T\backslash\alpha = T_2 \cup \cdots \cup T_m$. Then for any $n \in \mathbb{N}$, there is an exact triangle*

$$\begin{array}{ccc} SFH(-M_T,-\Gamma_n) & \longrightarrow & SFH(-M_T,-\Gamma_{n+1}) \\ & {\scriptstyle G_n}\nwarrow \quad \swarrow{\scriptstyle F_{n+1}} & \\ & SFH(-M_{T'},-\gamma_{T'}) & \end{array} \tag{A.2.7}$$

The map F_{n+1} is induced by the contact 2-handle attachment along the meridian of α. Furthermore, we have commutative diagrams related to $\psi^n_{+,n+1}$ and $\psi^n_{-,n+1}$, respectively

$$\begin{array}{ccc} SFH(-M_T,-\Gamma_n) & \xrightarrow{\psi^n_{\pm,n+1}} & SFH(-M_T,-\Gamma_{n+1}) \\ & {\scriptstyle G_n}\nwarrow \quad \nearrow{\scriptstyle G_{n+1}} & \\ & SFH(-M_{T'},-\gamma_{T'}) & \end{array}$$

and

$$\begin{array}{ccc} SFH(-M_T,-\Gamma_n) & \xrightarrow{\psi^n_{\pm,n+1}} & SFH(-M_T,-\Gamma_{n+1}) \\ & {\scriptstyle F_n}\searrow \quad \swarrow{\scriptstyle F_{n+1}} & \\ & SFH(-M_{T'},-\gamma_{T'}) & \end{array}$$

Proof. It follows from the proof of Lemma 3.1.8. □